The Divergence of a Vector Field

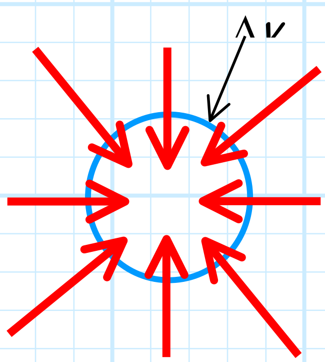
The **mathematical** definition of divergence is:

$$\nabla \cdot \mathbf{A}(\bar{r}) = \lim_{\Delta v \to 0} \frac{\oint\!\!\oint_S \mathbf{A}(\bar{r}) \cdot \overline{ds}}{\Delta v}$$

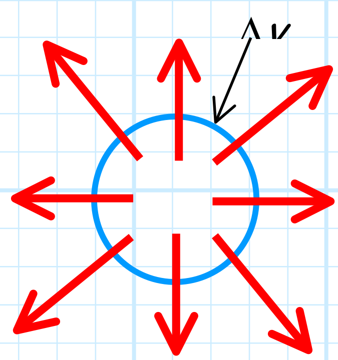
where the surface S is a **closed** surface that **completely** surrounds a **very small** volume Δv at point $\bar{r}$, and where $\overline{ds}$ points **outward** from the closed surface.

From the definition of surface integral, we see that divergence basically indicates the amount of vector field $\mathbf{A}(\bar{r})$ that is **converging to**, or **diverging from**, a given point.

For example, consider these vector fields in the region of a **specific point**:



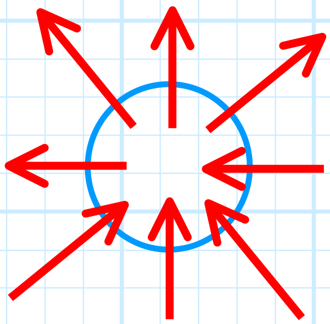
$\nabla \cdot \mathbf{A}(\bar{r}) < 0$



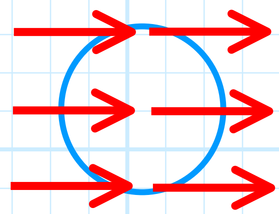
$\nabla \cdot \mathbf{A}(\bar{r}) > 0$

The field on the left is **converging** to a point, and therefore the divergence of the vector field at that point is **negative**. Conversely, the vector field on the right is **diverging** from a point. As a result, the divergence of the vector field at that point is **greater than zero**.

Consider some **other** vector fields in the region of a specific point:



$\nabla \cdot \mathbf{A}(\bar{r}) = 0$



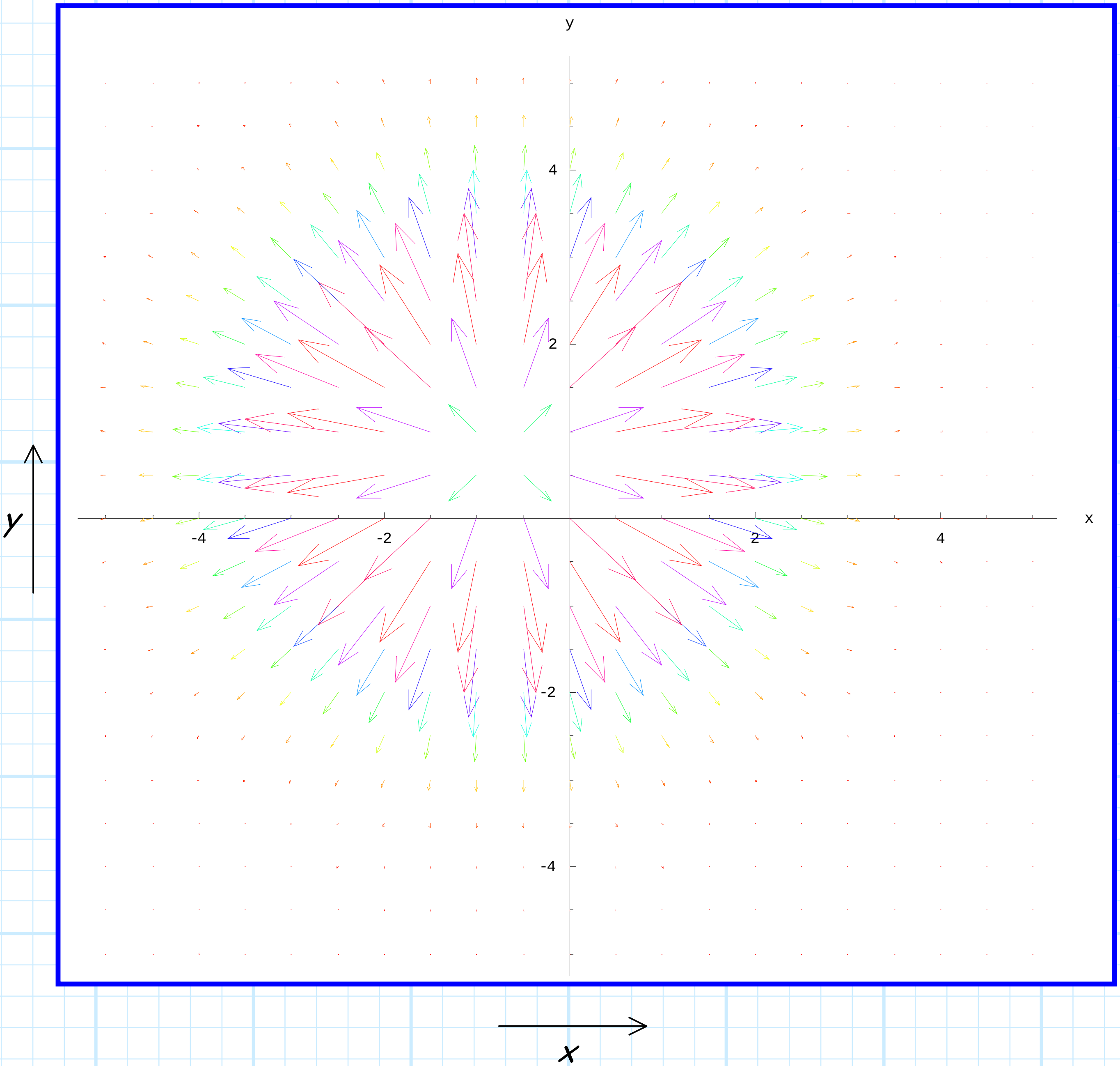
$\nabla \cdot \mathbf{A}(\bar{r}) = 0$

For each of these vector fields, the surface integral is **zero**. Over some portions of the surface, the normal component is positive, whereas on other portions, the normal component is negative. However, **integration** over the entire surface is equal to zero—the divergence of the vector field at this point is zero.

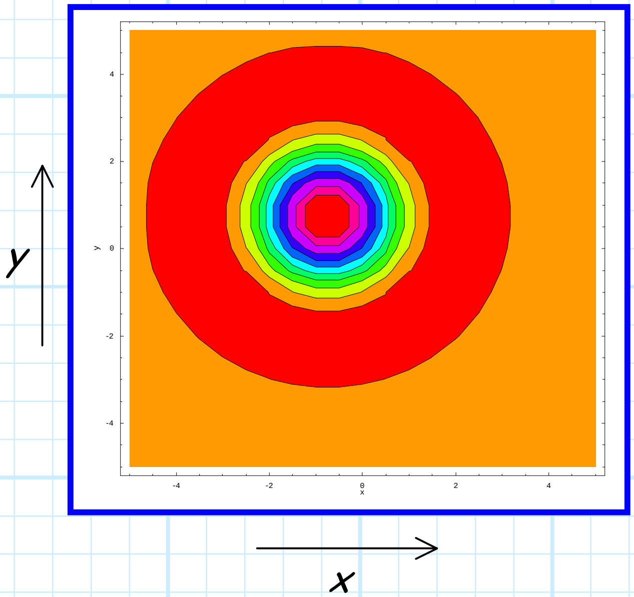
* **Generally**, the divergence of a vector field results in a scalar field (divergence) that is positive in some regions in space, negative other regions, and zero elsewhere.

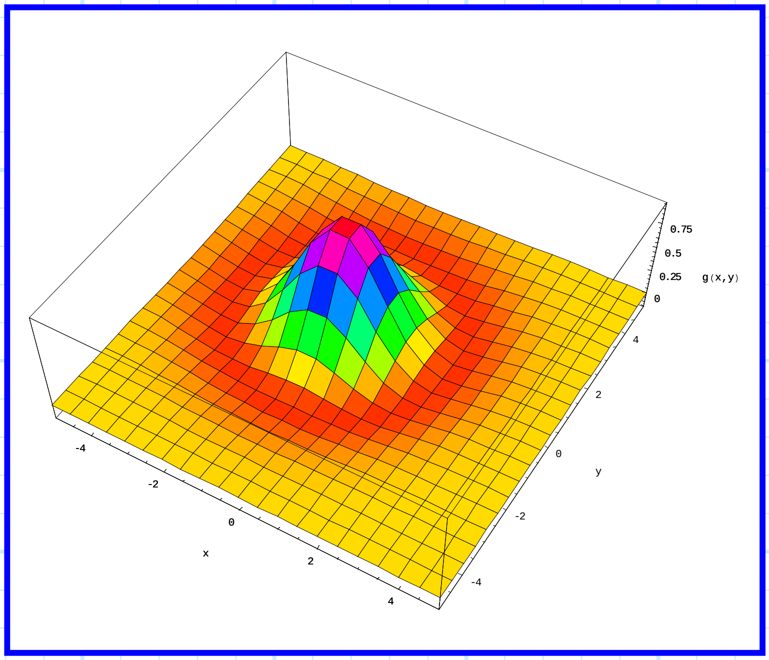
* For most **physical** problems, the divergence of a vector field provides a scalar field that represents the **sources** of the vector field.

For example, consider this two-dimensional vector field $\mathbf{A}(x,y)$, plotted on the x,y plane:



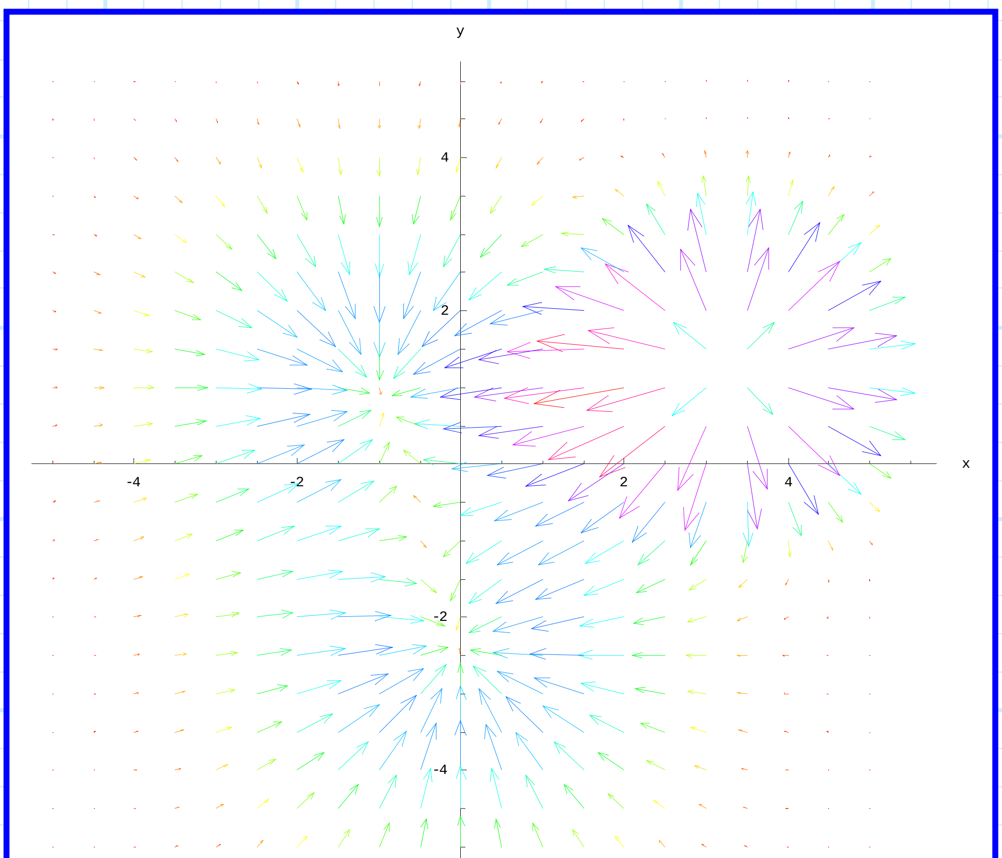
We can take the divergence of this vector field, resulting in the scalar field $g(x,y) = \nabla \cdot \mathbf{A}(x,y)$. Plotting this scalar function on the x,y plane:



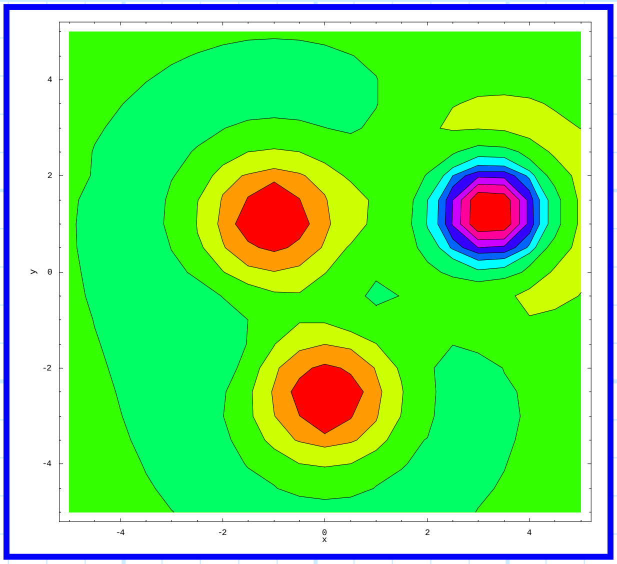


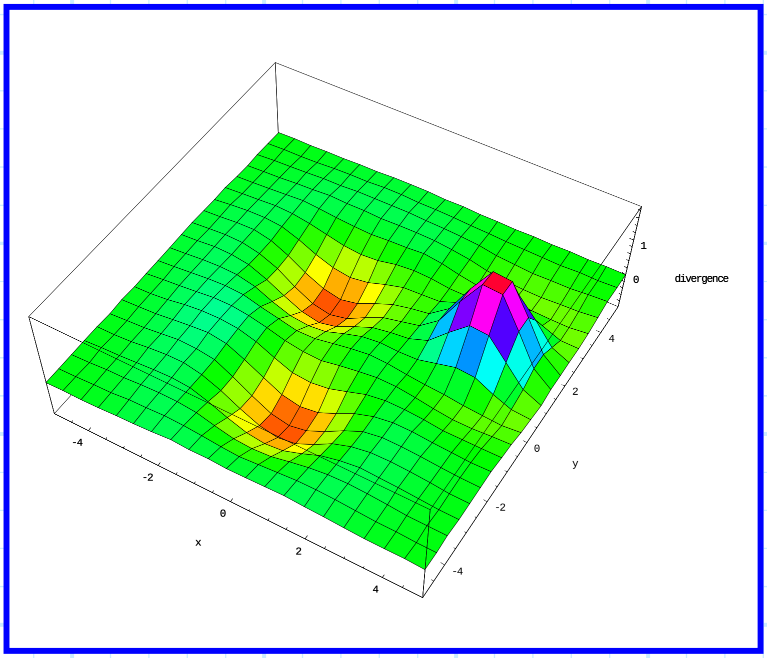
Both plots indicate that the divergence is largest in the vicinity of point x=-1, y=1. However, notice that the value of g(x,y) is non-zero (both positive and negative) for most points (x,y).

Consider now this vector field:

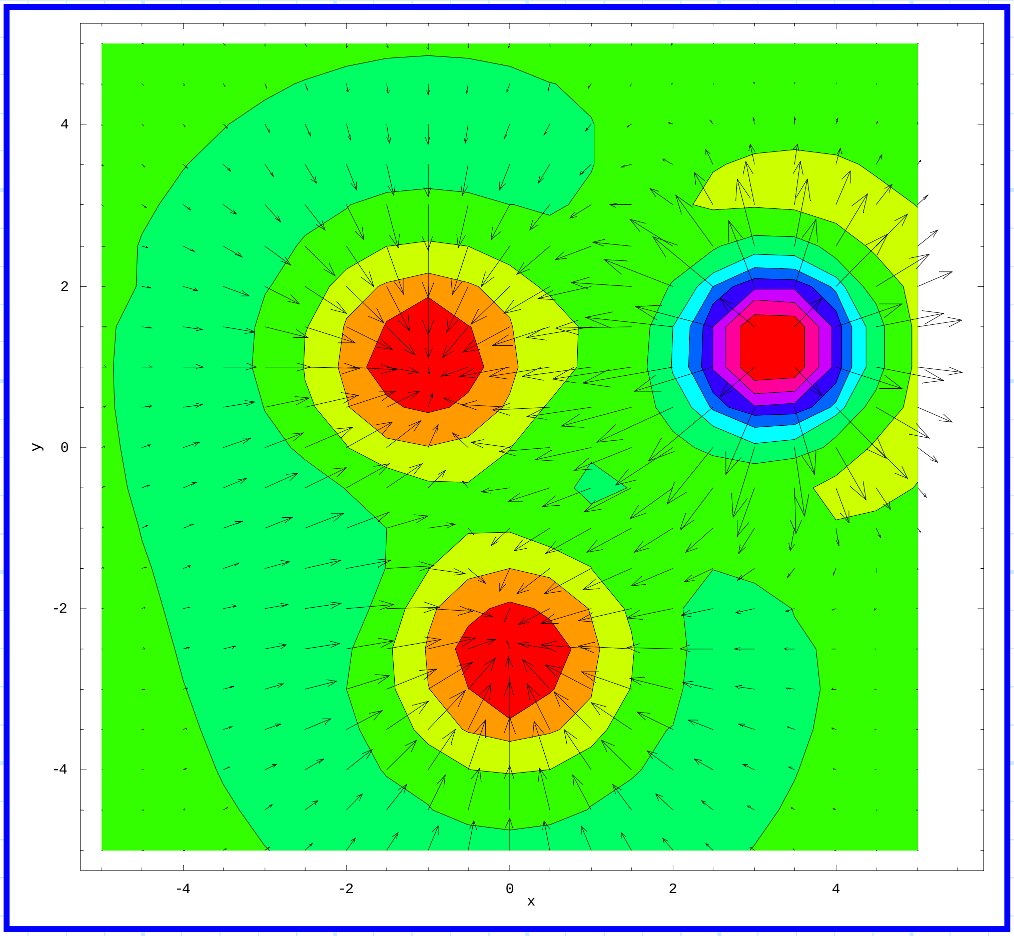


The **divergence** of this vector field is the **scalar** field:





Combining the vector field and scalar field plots, we can examine the **relationship** between each:



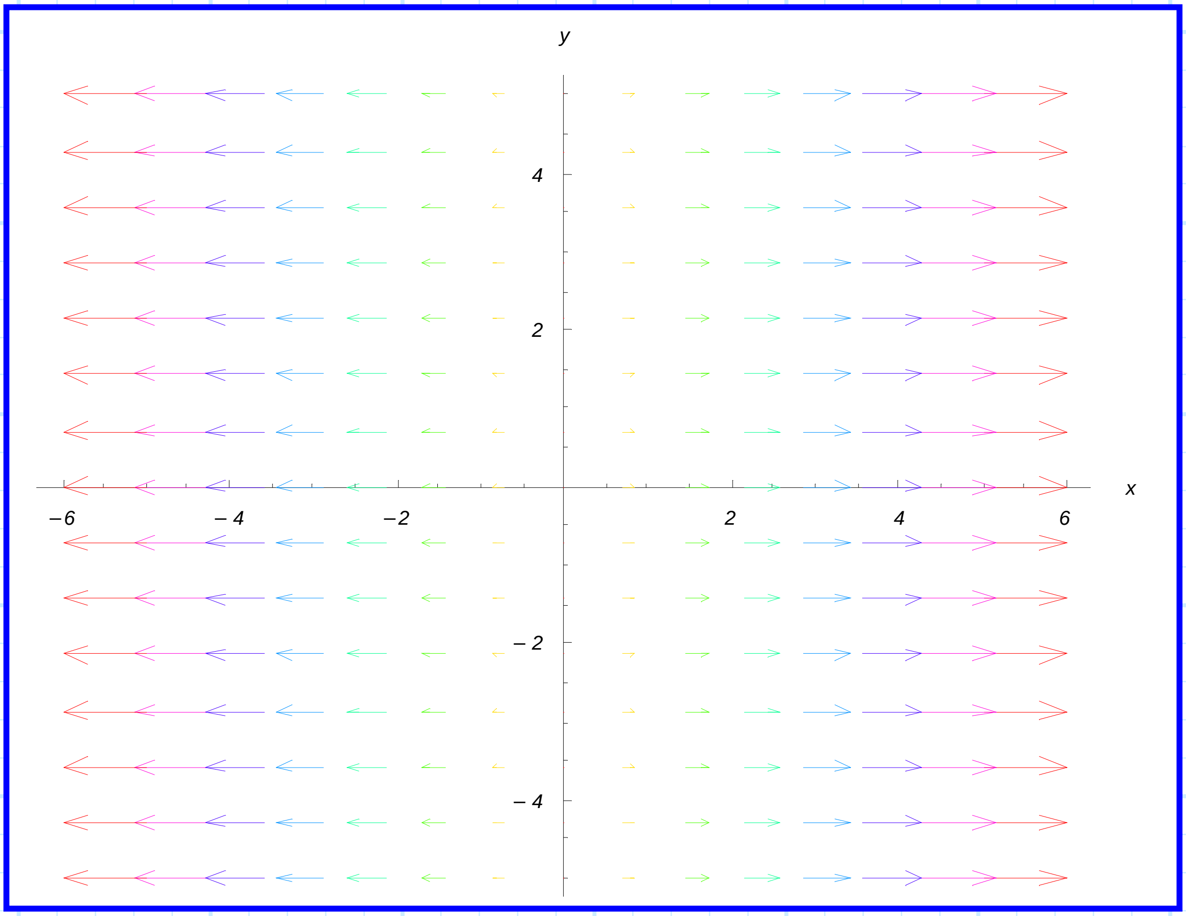
Look closely! Although the relationship between the scalar field and the vector field may appear at first to be the **same** as with the **gradient** operator, the two relationships are **very** different.

Remember:

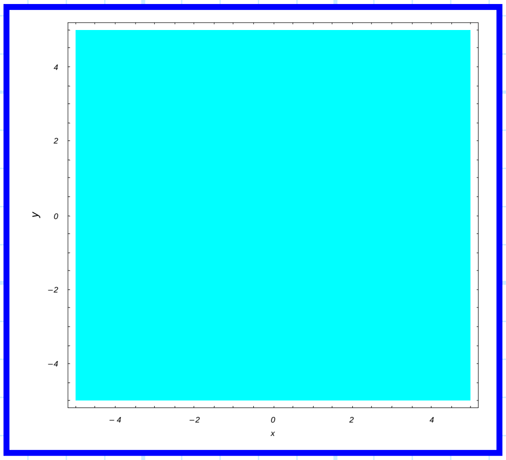
a) **gradient** produces a **vector** field that indicates the change in the original **scalar** field, whereas:

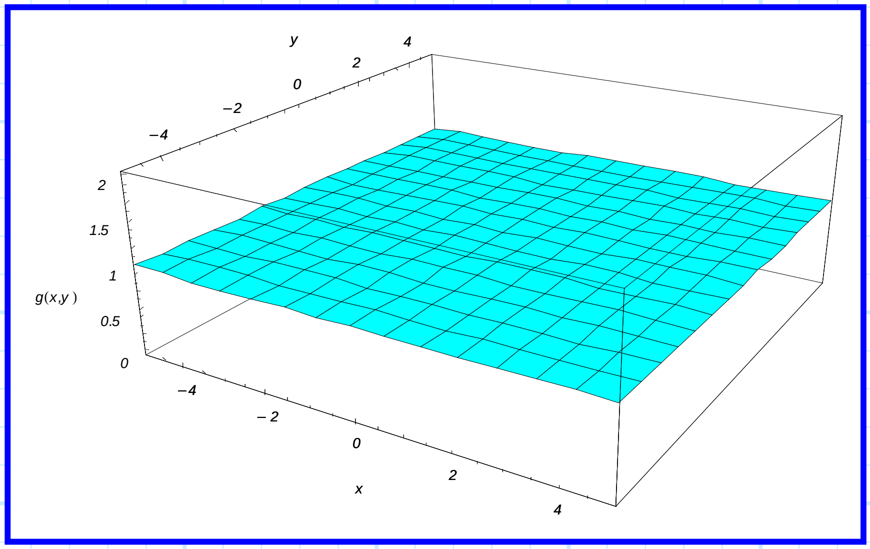
b) **divergence** produces a **scalar** field that indicates some change (i.e., divergence or convergence) of the original **vector** field.

The divergence of **this** vector field is interesting—it steadily increases as we move away from the *y*-axis.

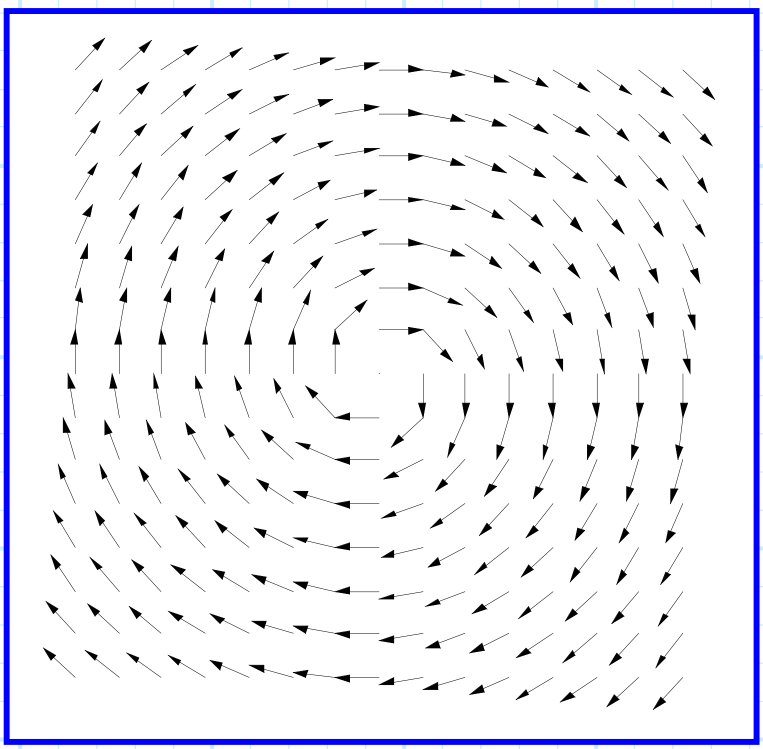


Yet, the divergence of this vector field produces a scalar field equal to one—**everywhere** (i.e., a **constant** scalar field)!

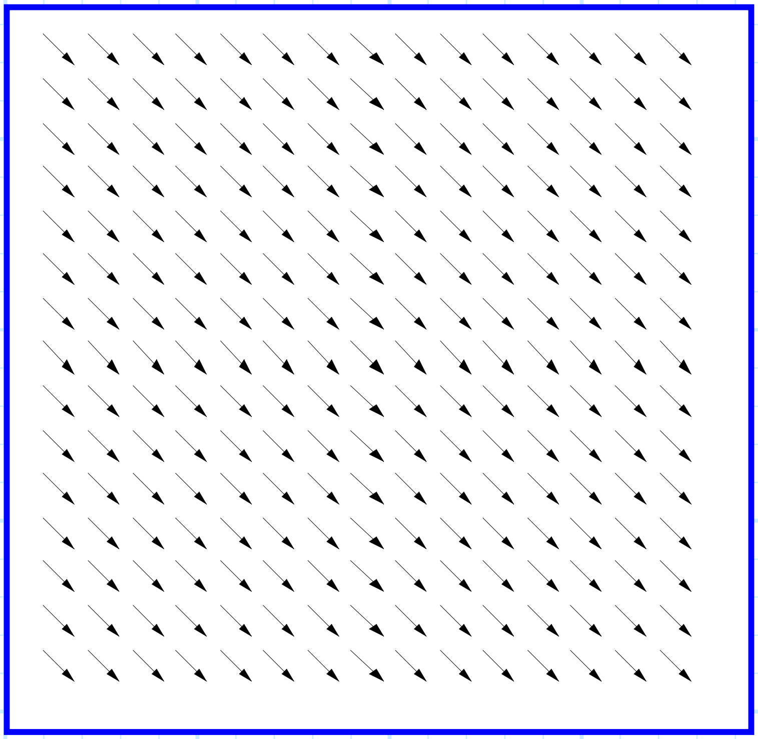




Likewise, note the divergence of these vector fields—it is **zero** at all points (x,y);



$$\boxed{\nabla \cdot \mathbf{A}(x,y) = 0}$$



$$\nabla \cdot \mathbf{A}(x,y) = 0$$

Although the examples we have examined here were all **two**-dimensional, keep in mind that both the original vector field, as well as the scalar field produced by divergence, will typically be **three-dimensional**!