EECS 863 HW 2 S08

1. Show that $R = QP$ is stochastic if Q and P are both stochastic.

Each element of R is $r_{ij} = \sum_k q_{ik} p_{kj}$

Since $q_{ik} \geq 0$ and $p_{kj} \geq 0 \Rightarrow \boxed{r_{ij} \geq 0}$

Also, sum over row i of R is:

$$\sum_j r_{ij} = \sum_j \sum_k q_{ik} p_{kj} = \sum_k q_{ik} \sum_j p_{kj}$$
$$= \sum_k q_{ik} (1) = \boxed{1 = \sum_j r_{ij}}$$

Since $\sum_j p_{kj} = 1$ and $\sum_k q_{ik} = 1$ (Rows sum to 1).

Conclude: R is stochastic.

Now for P stochastic $\Rightarrow P^n$ stochastic. $n \geq 1$

True for $n=1$ by assumption.

Induction proof for $n \geq 2$:

Initial step ($n=2$):

$\underline{P}^2 = \underline{P} \cdot \underline{P}$, so apply previous with $Q = P$.

Induction ($n \geq 3$):

Assume $\underline{P}^n$ stochastic, show $\underline{P}^{n+1}$ stochastic.

$\underline{P}^{n+1} = \underline{P}^n \cdot \underline{P}$ apply above with $Q = \underline{P}^n$

$\Rightarrow P^{n+1}$ stochastic

Conclude: $\underline{P}$ stochastic $\Rightarrow \underline{P}^n$ stochastic for $n \geq 1$.

2. $\underline{P} = \begin{bmatrix} 0 & 0.4 & 0.6 \\ 0.1 & 0.2 & 0.7 \\ 0.5 & 0 & 0.5 \end{bmatrix}$ $\pi(0) = [0.1 \quad 0.5 \quad 0.4]$

$e = [0.2 \quad 0.5 \quad 0.8]$.

$\pi(n) = \pi(n-1)\underline{P}$ $\quad Pr[\text{emission at time } n] = \pi(n) e^T$

Use Matlab.

$\pi(1) = \pi(0)\underline{P} = [0.25 \quad 0.14 \quad 0.61]$

$Pr[\text{emit}] = \pi(1) e^T = 0.608.$

$\pi(2) = \pi(1)\underline{P} = \pi(0)\underline{P}^2 = [0.319 \quad 0.128 \quad 0.553]$.

$Pr[\text{emit}] = \pi(2) e^T = 0.5702$

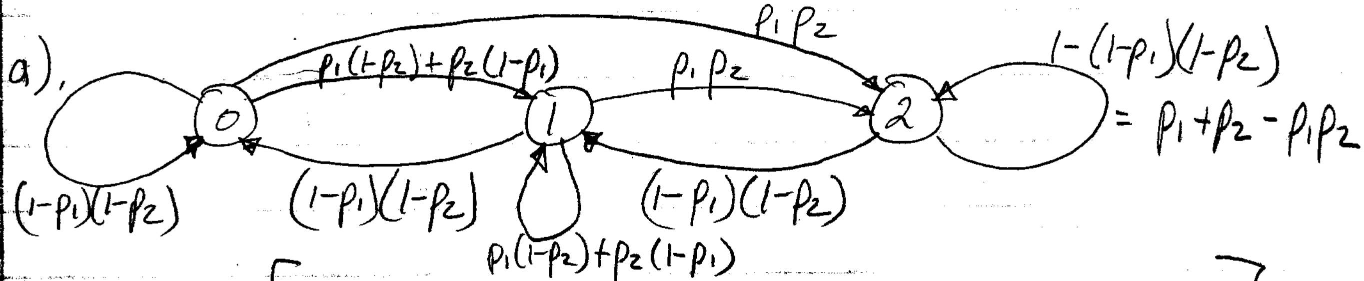
$\pi(3) = [0.2893 \quad 0.1532 \quad 0.5575]$ $\quad Pr[\text{emit}] = 0.5805$

$\pi(4) = [0.2941 \quad 0.1464 \quad 0.5596]$ $\quad Pr[\text{emit}] = 0.5797$

$\pi(5) = [0.2944 \quad 0.1469 \quad 0.5587]$ $\quad Pr[\text{emit}] = 0.5793.$

$\pi(100) = [0.2941 \quad 0.1471 \quad 0.5588]$ $\quad Pr[\text{emit}] = 0.5794$

3. a)



$$P = \begin{bmatrix} (1-p_1)(1-p_2) & p_1(1-p_2)+p_2(1-p_1) & p_1 p_2 \\ (1-p_1)(1-p_2) & p_1(1-p_2)+p_2(1-p_1) & p_1 p_2 \\ 0 & (1-p_1)(1-p_2) & 1-(1-p_1)(1-p_2) \end{bmatrix}$$

b). 2 at time t, less 1 departure leaves 1 in buffer. So there would have to be 2 arrivals.

Prob $= p_1 p_2$

c) 3 possible paths: $0\to2\to2$ or $0\to0\to2$ or $0\to1\to2$

$$\text{Prob} = p_1p_2\left[1-(1-p_1)(1-p_2)\right] + \left[(1-p_1)(1-p_2)\right]p_1p_2 + \left[p_1(1-p_2)+p_2(1-p_1)\right]p_1p_2$$
$$= p_1p_2\left[1+p_1(1-p_2)+p_2(1-p_1)\right] = p_1p_2\left[1+p_1+p_2-2p_1p_2\right]$$

4. $Pr[X(n)=i, X(n+m+l)=j]$
$= \sum_k Pr[X(n)=i, X(n+m)=k, X(n+m+l)=j]$.

$\underline{LHS} = Pr[X(n)=i] \cdot Pr[X(n+m+l)=j \mid X(n)=i]$.
$= \pi_i(n)\, p_{ij}(m+l)$.

From conditional prob chain rule,
$\underline{RHS} = \sum_k Pr[X(n)=i] \cdot Pr[X(n+m)=k \mid X(n)=i] \cdot$
$Pr[X(n+m+l)=j \mid X(n+m)=k, X(n)=i]$.

By Markov property,
last prob. is equal to $Pr[X(n+m+l)=j \mid X(n+m)=k]$.

so,
$\underline{RHS} = \sum_k \pi_i(n) \cdot p_{ik}(m) \cdot p_{kj}(l)$.

So, $\cancel{\pi_i(n)}\, p_{ij}(m+l) = \sum_k \cancel{\pi_i(n)}\, p_{ik}(m)\, p_{kj}(l)$.

or $p_{ij}(m+l) = \sum_k p_{ik}(m)\, p_{kj}(l)$.

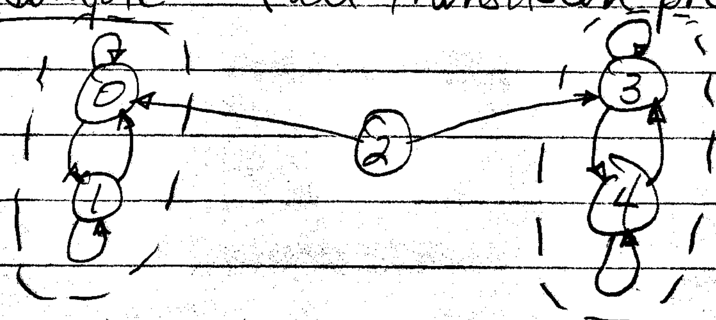
5. Let π be soln to $\pi = \pi P$ ad let $\pi(0) = \pi$

Then $\pi(1) = \pi(0)\underline{P} = \pi(0) = \pi$
$\pi(2) = \pi(1)\underline{P} = \pi(0)\underline{P} = \pi(0) = \pi$
$\vdots$

So $\pi(n) = \pi \;\; \forall n \Rightarrow$ marginal pmf is constant.
From previous C-K derivation (previous problem)
$Pr[X(n)=i, X(n+m)=k, X(n+m+l)=j]$
$= \pi_i(n)\, p_{ik}(m)\, p_{kj}(l) = \pi_i \cdot p_{ik}(m)\, p_{kj}(l)$
which depends only on time <u>differences</u> m & l.

6. In terms of state diagrams, if you can draw the state diagram in such a way that some subset of states can be encircled with no transitions leaving this subset, then the subset is absorbing and the DTMC is <u>not</u> irreducible.

Example: (all transition probs are 0.5).



Two sets of absorbing states.

7. Assume $p_{ii} > 0$ for some i.
Then, regardless of the partition, state i must be in one of the subsets — call the subset containing state i S_j. Since state i is reachable, $Pr[X(n+1) \in S_j \mid X(n) \in S_j] > 0$ and so the Prob. of transitioning from S_j to <u>any</u> other subset must be strictly < 1.0 and the chain cannot be periodic.

5. To show that a finite-state, irreducible DTMC is recurrent, consider arbitrary state i and consider the partition $S_0 = \{i\}$ and $S_1 = \{\text{all other states}\}$. Then $Pr[\text{never return to } i] = Pr[\text{transition to } S_1 \text{ at first step}] \cdot Pr[\text{stay in } S_1 \text{ forever}]$.
But $Pr[\text{stay in } S_1 \text{ forever}] = 0$ because (from irreducible) the prob. of transitioning from <u>every</u> state in S_1 to state i must be non-zero in some <u>finite</u> number of steps. Finite size of S_1 ensures $\mu_i < \infty$ (proof not given)