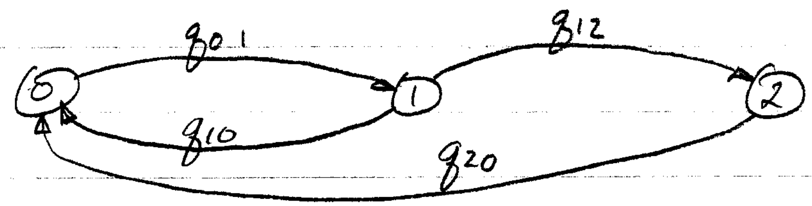
863 HW3 S'08

1. a). $Q = \begin{bmatrix} -q_{01} & q_{01} & 0 \\ q_{10} & -(q_{10}+q_{12}) & q_{12} \\ q_{20} & 0 & -q_{20} \end{bmatrix}$

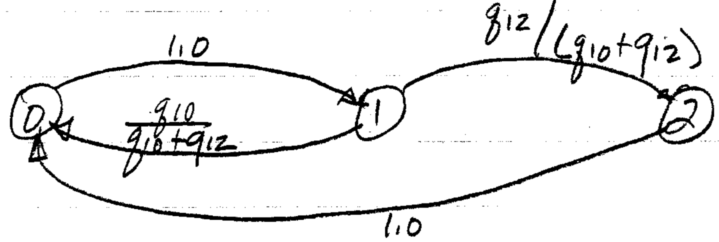


b). $\frac{d}{d\tau} \pi_0(\tau) = -q_{01}\pi_0(\tau) + q_{10}\pi_1(\tau) + q_{20}\pi_2(\tau)$.

$d/d\tau\ \pi_1(\tau) = q_{01}\pi_0(\tau) - (q_{10}+q_{12})\pi_1(\tau)$

$d/d\tau\ \pi_2(\tau) = q_{12}\pi_1(\tau) - q_{20}\pi_2(\tau)$

c).



d). Yes, because embedded chain is irreducible and recurrent.

Solve $0 = \pi Q$ with $\sum \pi_i = 1$

e.g. solve: $[1\ 0\ 0] = [\pi_0\ \pi_1\ \pi_2] \begin{bmatrix} 1 & q_{01} & 0 \\ 1 & -(q_{10}+q_{12}) & q_{12} \\ 1 & 0 & -q_{20} \end{bmatrix}$

Solution:

$\pi_0 = \dfrac{q_{20}q_{10} + q_{20}q_{12}}{s}$

$\pi_1 = q_{20}q_{01}/s$

$\pi_2 = q_{12}q_{01}/s$

$s = q_{20}q_{10} + q_{20}q_{12} + q_{20}q_{10} + q_{12}q_{01}$

1. e). Mean sojourn time in state 0 : q_{01}^{-1}

" " " " " 1 : $(q_{10}+q_{12})^{-1}$

" " " " " 2 : q_{20}^{-1}

f). q_{20} : failure rate
q_{01} : repair rate
q_{12} : successful test rate
q_{10} : failed test rate.

Note (example): since $\frac{1}{q_{01}}$ is mean time of a unit in repair facility, the facility can repair items at an avg rate of q_{01} (repair rate

2. $p_{ij}(\tau+dt) = \sum_k p_{ik}(dt)\, p_{kj}(\tau) = p_{ii}(dt)\, p_{ij}(\tau) + \sum_{k \neq i} p_{ik}(dt)\, p_{kj}(\tau)$

$$= [1+q_{ii}dt]\, p_{ij}(\tau) + \sum_{k\neq i} q_{ik}\, dt\, p_{kj}(\tau)$$

or

$$\frac{p_{ij}(\tau+dt)-p_{ij}(\tau)}{dt} = \sum_k q_{ik}\, p_{kj}(\tau)$$

$dt \to 0$

$$\frac{d}{d\tau} p_{ij}(\tau) = \sum_k q_{ik}\, p_{kj}(\tau)$$

Matrix form: $\frac{d}{d\tau} \underline{P}(\tau) = Q\, \underline{P}(\tau)$ where $\underline{P}(\tau)$ in this form is a column vector.

Surprisingly, this is an equivalent set of equations to $d/d\tau\, \underline{P}(\tau) = \underline{P}(\tau)\, Q$ in the sense of producing the same $\underline{P}(\tau)$.

3. Global Balance: (follow DTMC derivation)

$$\sum_{i \in S} \pi_j q_{ji} = \sum_{i \in S} \pi_i q_{ij} \quad \forall j$$

Partition states S into (S_1, S_2), sum over $j \in S_1$

$$\sum_{j \in S_1} \pi_j \sum_{i \in S} q_{ji} = \sum_{i \in S} \pi_i \sum_{j \in S_1} q_{ij}$$

$$\sum_{j \in S_1} \pi_j \left[\sum_{i \in S_1} q_{ji} + \sum_{i \in S_2} q_{ji} \right] = \left[\sum_{i \in S_1} \pi_i + \sum_{i \in S_2} \pi_i \right] \sum_{j \in S_1} q_{ij}$$

$$\sum_{j \in S_1} \pi_j \sum_{i \in S_2} q_{ji} = \sum_{i \in S_2} \pi_i \sum_{j \in S_1} q_{ij}$$

Rate out of S_1 = Rate into S_1

4. (3.5) $1/\lambda = 100ms = 0.1s \Rightarrow \lambda = 10$ msgs/sec
In interval [0,1], 10 msgs are <u>actually</u> generated.

a) Arrivals in [1, 1.5] are <u>indep</u> of arrivals in [0,1], so they are poisson with $\tau = 0.5$:

$$A_n(\tau) = \frac{(\lambda\tau)^n e^{-\lambda\tau}}{n!}$$

With $\lambda=10$ and $\tau=0.5$, $A_n(0.5) = \dfrac{5^n e^{-5}}{n!}$

$$\Pr\{5 \text{ or more in } [1,1.5]\} = 1 - \sum_{n=0}^{4} A_n(0.5) = 0.5595$$

b) Arrivals in [0.75, 1] <u>not</u> indep of arrivals in [0,1]
However, arrivals in [0, 0.75] <u>are</u> indep of arrivals in [0.75, 1] and we know the <u>sum</u> must be 10.
This gives 6 possibilities: for each i from 6 to 10:
i arrivals in [0.75, 1] and $10-i$ in [0, 0.75]

$$\text{Prob} = \frac{\sum_{i=5}^{10} A_i(0.25)\, A_{10-i}(0.75)}{A_{10}(1)} \quad \text{with } \lambda = 10$$

condition on 10 arrivals in [0,1].

$$= \frac{0.009774}{0.12511} = 0.0781$$

5. A exponential with $\lambda = 0.2$ arrivals/min

a). $1/\lambda = 5$ min

b) Due to memoryless property of exponential, you will still expect to wait 5 minutes.
And if, at the end of 5 minutes a taxi has still not shown up, your expected wait is still 5 minutes from <u>that</u> time!

6. a). $\binom{m}{k}(\lambda \Delta t)^k (1-\lambda \Delta t)^{m-k}$

Special form of Binomial.

b). $\binom{m}{k}\left(\lambda \frac{T}{m}\right)^k \left(1-\lambda\frac{T}{m}\right)^{m-k}$

$$= \frac{(m)(m-1)\cdots(m-k+1)}{k!}\left(\frac{\lambda T}{m-\lambda T}\right)^k\left(1-\frac{\lambda T}{m}\right)^m$$

As $m \to \infty$, approaches: $\frac{\cancel{m^k}}{k!}\frac{(\lambda T)^k}{\cancel{m^k}} e^{-\lambda T} = \frac{(\lambda T)^k e^{-(\lambda T)}}{k!}$

Alternate derivation of Poisson distribution for pure-birth, constant birth rate CTMC, at time T.

7. a). Since in state k, there are k particles, each of which is splitting (increasing population by 1) at rate λ, we have

$$1 \xrightarrow{\lambda} 2 \xrightarrow{2\lambda} 3 \xrightarrow{3\lambda} 4 \xrightarrow{4\lambda} \cdots \qquad Q = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & \\ 0 & -2\lambda & 2\lambda & 0 & \cdots \\ & \vdots & & & \end{bmatrix}$$

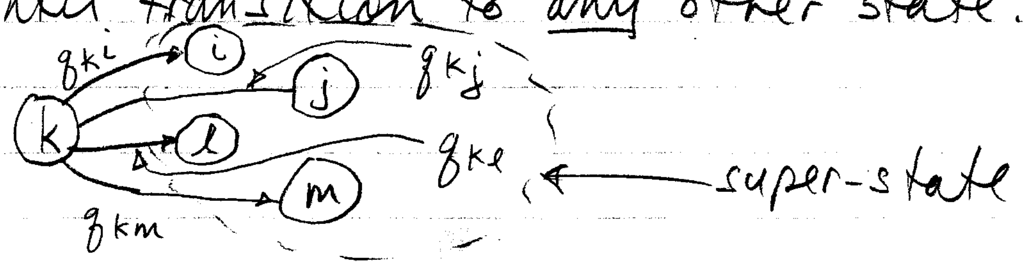
b). From general result:

$$\frac{d}{d\tau}\pi_1(\tau) = -\pi_1(\tau)\cdot\lambda$$

and $$\frac{d}{d\tau}\pi_k(\tau) = \pi_{k-1}(\tau)\cdot(k-1)\lambda - \pi_k(\tau)\cdot k\lambda \quad k=2,3,\ldots$$

c). Clearly transient: once it leaves a state, it never returns.

9. Following method for Poisson process, consider given state k and consider dist of time until transition to any other state.



Can create "super-state" as set of all these other states. The rate of transition into the "super-state" is

$$\lambda_k = \sum_{n \neq k} q_{kn}$$

Can now use derivation exactly as for Poisson case to see that the time until a transition from state k is exponential w/ param λ_k (mean $= 1/\lambda_k$).

8. Differs only in "direction" that the process runs (transitions to next lower neighbor as opposed to next higher neighbor) so, pmf of # deaths in τ is Poisson with parameter μ (and τ)

$$D_n(\tau) = \frac{(\mu\tau)^n e^{-\mu\tau}}{n!} \qquad n = 0, 1, \ldots$$

and inter-death time is exp. w/ param μ

$$f_{T_d}(\tau) = \mu e^{-\mu\tau} \qquad \tau > 0.$$