2. a). pdf of \( t \): \( f_{t_j}(t) = \lambda e^{-\lambda t} \).

b). Prob that state \( i_{j+1} \) follows state \( i_j \):
\[
\frac{g_{i_j i_{j+1}}}{\sum_{i} g_{i_j i}} = \frac{g_{i_j i_{j+1}}}{\lambda i_j}.
\]

c). Prob that \( X(t) \) stays in \( i_m \) for at least \( t_m \):
\[
\int_{t_m}^{\infty} e^{-\lambda i_m t_m} dt = e^{-\lambda i_m t_m}.
\]
d). Joint pdf of \( t_1, t_2, \ldots t_m \):
\[
\Pi_i \Pi_{i_j} e^{-\lambda i_1 t_1} \cdot \frac{g_{i_1 i_2}}{\lambda i_2} \cdot \frac{g_{i_2 i_3}}{\lambda i_3} \cdots e^{-\lambda i_m t_m} = \Pi_i e^{-\lambda i_1 t_1} \cdot g_{i_1 i_2} \cdot \frac{g_{i_2 i_3}}{\lambda i_3} \cdots e^{-\lambda i_m t_m}.
\]
e). Parallel reasoning - result:
\[
\Pi_i \Pi_{i_j} e^{-\lambda i_1 t_1} \cdot \frac{g_{i_1 i_2}}{\lambda i_2} \cdot \frac{g_{i_2 i_3}}{\lambda i_3} \cdots e^{-\lambda i_m t_m} = \Pi_i e^{-\lambda i_1 t_1} \cdot g_{i_1 i_2} \cdot \frac{g_{i_2 i_3}}{\lambda i_3} \cdots e^{-\lambda i_m t_m}.
\]
f). Local balance:
\[
\Pi_i g_{i_1 i_2} = \Pi_i g_{i_2 i_1} \Rightarrow \Pi_i g_{i_2 i_3} = \Pi_i g_{i_3 i_2}.
\]
Multiply together: \( \Pi_i g_{i_1 i_2} \Pi_i g_{i_2 i_3} = \Pi_i g_{i_3 i_2} \Pi_i g_{i_2 i_1} \)
Then \( \Pi_i g_{i_3 i_4} = \Pi_i g_{i_4 i_3} \), multiply, cancel \( \Pi i_3 \):
\[
\Pi_i g_{i_1 i_2} g_{i_2 i_3} g_{i_3 i_4} = \Pi_i g_{i_4 i_3} g_{i_3 i_2} g_{i_2 i_1}.
\]
2. \( f(t) \) (cont.) continue repeatedly to get:

\[ \pi_1 q_{1,2} \cdots q_{m-1,m} = T_{m} q_{m-1,m-1} \cdots q_{1,2} \]

which implies that polfs are identical since all exponential factors are the same.

g) Joint dist of \( X(t_1), X(t_2) \cdots X(t_m) \) is identical to joint dist of \( X(-t_1), X(-t_2), \cdots X(-t_m) \)

and if the process \( X(t) \) is stationary, this is identical to joint dist of \( X(t-t_1), X(t-t_2), \cdots X(t-t_m) \)

\[ \Rightarrow X(t) \text{ reversible!} \]

\[ \text{1. } Y(t) = \mathbb{T}(\lambda(1-z)) = \int_0^\infty e^{-\lambda(1-z)t} f_T(t) dt \]

\[ \mathbb{E}[Y] = Y'(1) = \left. \int_0^\infty \lambda t e^{-\lambda(1-z)t} f_T(t) dt \right|_{z=1} = \lambda \mathbb{E}[T] \]

\[ a_1 \mathbb{E}[Y(Y-1)] = Y''(1) = \left. \int_0^\infty (\lambda t)^2 e^{-\lambda(1-z)t} f_T(t) dt \right|_{z=1} = \lambda^2 \mathbb{E}[T^2] \]

\[ \text{c. Pattern is clear: } \mathbb{E}[Y(Y-1)(Y-2)\cdots(Y-n+1)] = \lambda^n \mathbb{E}[T^n] \].
3. \((k+1)\)

a) \(M/G/1\) with finite queue?

Embed the MC at departure instants.

The fundamental problem is that we could not write one queue evolution equation that would be valid for all states, so in:

\[ y_{k+1} = y_k - u[y_{k+1}] + a_{k+1} \]

which is valid for all \(k\) and for all states \(y_k\).

This would destroy our MGF analysis. However, you could do a brute-force analysis for a given max number in system \(N\) and a given service time distribution, resulting in a particular, finite-state DT (embedded) Markov Chain, which could be solved with \(P = TP\).

b) \(M/G/1\) with infinite queue but \(>1\) server?

Again, try to embed the MC at departure instants. But there would have to be departures from any server, and so once again it would be impossible to write a queue evolution equation similar to

\[ y_{k+1} = y_k - u[y_{k+1}] + a_{k+1} \]

Since in this equation, \(a_{k+1}\) is the number of arrivals in the service time of the \((k+1)\)st customer. With multiple servers, it would have to be arrivals since last departure from any server, which we cant characterize.
4. With $A(1) = 1$, indeterminism comes from

$$1 - \frac{1}{Z} = \frac{Z-1}{Z-A(Z)}$$

So use L'Hopital's rule: $\lim_{Z \to 1} \frac{Z-1}{Z-A(Z)} = \lim_{Z \to 1} \frac{1}{1-A'(Z)} = \frac{1}{1-p}$.

So we have $1 = \Pi_0(1/p)$ or $\Pi_0 = 1 - p$.

5. $Y(Z) = A(Z) \cdot \frac{(1-p)(1-Z)}{A(Z)-Z}$. $A(Z)$ = L.T. of service time pdf

For $M/M/1$ : $f_\theta(t) = \mu e^{-\mu t}$

$$A(Z) = \int_0^\infty e^{-\lambda(1-Z)t} \mu e^{-\mu t} dt = \mu \left[ \frac{1}{\mu + \lambda - \lambda Z} \right] \left[ 0 - 1 \right]$$

$$A(Z) = \frac{\mu}{\mu + \lambda - \lambda Z}$$

and $A(Z)-Z = \frac{\mu - Z\mu + \lambda - \lambda Z}{\mu + \lambda - \lambda Z}$

Then $Y(Z) = \frac{\mu (1-p)(1-Z)}{\mu - Z\mu + \lambda - \lambda Z} = \frac{(1-p)(1-Z)}{(1-p)Z(1-Z)} = \frac{1-p}{1-Z}$

Taking the inverse transform:

$\Pi_n = (1-p) p^n \quad n = 0, 1, \ldots$ as expected.

7. $\mu = 2 \quad \lambda = 1/3 \implies \rho = 2/3 < 1$ (OK).

a. $M/M/1 : \bar{\tau} = \frac{1/\mu}{1-p} = \frac{2}{1/3} = 6$ time units.

b. $\bar{\tau} = \frac{1}{\mu} + \frac{\lambda (\mu^2 + 3\mu^2)}{2(1-p)} \quad \bar{B}^2 = 0$

$$= 2 + \frac{\frac{1}{3}(2^2+0)}{2(1-2/3)} = 2 + \frac{1/3}{2/3} = 4 \text{ time units}.$$
7. (c) \( f_B(b) = \frac{1}{3} \) for \( 1 < b < 3 \)

\[ E[B] = \frac{1}{3} \int_1^3 \frac{1}{3} b^2 \, db = \frac{1}{3} \left[ \frac{1}{4} b^4 \right]_1^3 = \frac{1}{3} \left( \frac{81}{4} - \frac{1}{4} \right) = \frac{24}{6} = \frac{13}{3} \]

\[ E[B^2] = \frac{1}{3} \int_1^3 b^3 \, db = \frac{1}{3} \left[ \frac{1}{4} b^4 \right]_1^3 = \frac{1}{3} \left( \frac{81}{4} - \frac{1}{4} \right) = \frac{24}{6} = \frac{13}{3} \]

\[ \bar{T} = \frac{1}{\mu} + \frac{\lambda E[B^2]}{2(1 - \rho)} = 2 + \frac{(1/3)(13/3)}{2(1 - 13/3)} = 2 + \frac{13}{6} = 4.17 \text{ time units} \]

(d) \( f_B(b) = k(b-2)^2 \) for \( 1 < b < 3 \)

\[ \int_1^3 k(b-2)^2 \, db = k \left( \frac{(b-2)^3}{3} \right)_1^3 = k \left( \frac{1}{3} - \frac{-1}{3} \right) = k \frac{2}{3} = 1 \]

\[ \Rightarrow k = \frac{3}{2} \]

By symmetry, \( E[B] = 2 = \frac{1}{\lambda} \)

\[ E[B^2] = \int_1^3 k(b-2)^2 \, db = \frac{3}{2} \left( \frac{1}{6} \right) (b-2)^3 \left|_1^3 \right. = 0.3 \, (1-(-1)) \]

\[ \bar{T} = 2 + \frac{1/6 (2^2 + 0.6)}{2(1 - 13/3)} = 4.30 \text{ time units} \]

(e) \( f_B(b) = \frac{1}{2} \delta(b-1) + \frac{1}{2} \delta(b-3) \) for \( 1 < b < 3 \)

\[ E[B] = 2 = \frac{1}{\lambda} \]

\[ E[B^2] = \int_{-\infty}^{\infty} \frac{1}{2} b^2 \delta(b-1) + \frac{1}{2} b^2 \delta(b-3) \, db = \frac{1}{2} \left[ 1^2 + 3^2 \right] = 5 \]

\[ \bar{T} = 2 + \frac{(1/3)(5)}{2(1 - 13/3)} = 2 + \frac{5}{6} = 4.50 \text{ time units} \]

7. Service time dist

<table>
<thead>
<tr>
<th>f(b)</th>
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<tbody>
<tr>
<td>0</td>
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<tr>
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<tr>
<td>3</td>
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<tr>
<td>4</td>
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<tr>
<td>5</td>
</tr>
</tbody>
</table>

- **Mean Time in System**: 4 units - min variance
- **Mean Time in System**: 4.17
- **Mean Time in System**: 4.3
- **Mean Time in System**: 4.5 - Max variance given limits
- **Mean Time in System**: 6.0
\[ (a) \quad E(n) = \frac{d\mu(x)}{dx} \bigg|_{x=1} = \frac{d}{dx} \left( \frac{1-p}{1-p^2} \right) \bigg|_{x=1} = \frac{(1-p^2)(1-p) - (1-p)(1-p)}{(1-p^2)^2} \bigg|_{x=1} = \frac{(1-p)(p)}{(1-p)^3} = \frac{1-p}{1-p} = E(n) \]

\[ (b) \quad \frac{d^2\mu(x)}{dx^2} \bigg|_{x=1} = \frac{d}{dx} \left( \frac{-p^2-p}{(1-p^2)^2} \right) \bigg|_{x=1} = \frac{+(p^2-p)2(1-p^2)(1-p)}{(1-p^2)^4} \bigg|_{x=1} = \frac{2p^2(1-p)}{(1-p)^3} \]

\[ = \frac{2p^2}{(1-p)^2} = E(n^2) - E(n) \]

So:

\[ E(n^2) = \frac{2p^2}{(1-p)^2} + \frac{p}{1-p} \]

\[ = \frac{2p^2 + p(1-p)}{(1-p)^2} \]

\[ = \frac{2p^2 + p - p^2}{(1-p)^2} = \frac{p^2 + p}{(1-p)^2} \]

\[ 6h^2 = E(n^2) - (E(n))^2 = \frac{p^2 + p}{1-p^2} - \frac{p^2}{(1-p)^2} = \frac{p}{(1-p)^2} \]