EECS 863: HW 8 i.508

2. a). pdf of t_ℓ : $f_{t_\ell}(t_\ell) = \lambda_{i_\ell} e^{-\lambda_{i_\ell} t_\ell}$.

b). Prob that state $i_{\ell+1}$ follows state i_ℓ:

$$\frac{q_{i_\ell i_{\ell+1}}}{\sum_{j \neq i_\ell} q_{i_\ell j}} = \frac{q_{i_\ell i_{\ell+1}}}{\lambda_{i_\ell}}.$$

c). Prob that $X(t)$ stays in i_m for at least t_m:

$$\int_{t_m}^{\infty} \lambda_{i_m} e^{-\lambda_{i_m} t} dt = e^{-\lambda_{i_m} t_m}$$

d). Joint pdf of $t_1, t_2, \ldots t_m$:

$$\pi_{i_1} \cancel{\lambda_{i_1}} e^{-\lambda_{i_1} t_1} \cdot \frac{q_{i_1 i_2}}{\cancel{\lambda_{i_1}}} \cdot \cancel{\lambda_{i_2}} e^{-\lambda_{i_2} t_2} \cdot \frac{q_{i_2 i_3}}{\cancel{\lambda_{i_2}}} \cdots \cdot e^{-\lambda_{i_m} t_m}$$

$$= \pi_{i_1} e^{-\lambda_{i_1} t_1} q_{i_1 i_2} e^{-\lambda_{i_2} t_2} \cdot \cdots \cdot q_{i_{m-1} i_m} e^{-\lambda_{i_m} t_m}.$$

e). Parallel reasoning - result:

$$\pi_{i_m} e^{-\lambda_{i_m} t_m} \cdot q_{i_m i_{m-1}} e^{-\lambda_{i_{m-1}} t_{m-1}} \cdot \cdots \cdot q_{i_2 i_1} e^{-\lambda_{i_1} t_1}$$

f) Local balance:

$$\pi_{i_1} q_{i_1 i_2} = \pi_{i_2} q_{i_2 i_1}, \quad \pi_{i_2} q_{i_2 i_3} = \pi_{i_3} q_{i_3 i_2}.$$

Multiply together: $\pi_{i_1} q_{i_1 i_2} \cancel{\pi_{i_2}} q_{i_2 i_3} = \pi_{i_3} q_{i_3 i_2} \cancel{\pi_{i_2}} q_{i_2 i_1}$

Then $\pi_{i_3} q_{i_3 i_4} = \pi_{i_4} q_{i_4 i_3}$, multiply, cancel π_{i_3}:

$$\pi_{i_1} q_{i_1 i_2} q_{i_2 i_3} q_{i_3 i_4} = \pi_{i_4} q_{i_4 i_3} q_{i_3 i_2} q_{i_2 i_1}$$

2. f) (con't) continue repeatedly to get:

$$\pi_{i_1} q_{i_1 i_2} \cdots q_{i_{m-1} i_m} = \pi_{i_m} q_{i_m i_{m-1}} \cdots q_{i_2 i_1}$$

which implies that pdf's are identical since all exponential factors are the same.

g). Joint dist of $X(t_1), X(t_2) \cdots X(t_m)$ is identical to joint dist of $X(-t_1), X(-t_2), \cdots X(-t_m)$ and if the process $X(t)$ is stationary, this is identical to joint dist of $X(\tau-t_1), X(\tau-t_2), \cdots X(\tau-t_m)$
$\Rightarrow X(t)$ reversible!

r. $Y(z) = T(\lambda(1-z)) = \int_0^\infty e^{-\lambda(1-z)t} f_T(t)\,dt$

$E[Y] = Y'(1) = \int_0^\infty \lambda t e^{-\lambda(1-z)t} f_T(t)\,dt \Big|_{z=1} = \lambda E[T]$.

a) $E[Y(Y-1)] = Y''(1) = \int_0^\infty (\lambda t)^2 e^{-\lambda(1-z)t} f_T(t)\,dt \Big|_{z=1} = \lambda^2 E[T^2]$.

c) Pattern is clear: $E[Y(Y-1)(Y-2)\cdots(Y-n+1)] = \lambda^n E[T^n]$.

3. (6.1)

a) M/G/1 with finite queue?
Embed the MC at departure instants.
The fundamental problem is that we could not write one queue evolution equation that would be valid for all states, as in:

$$y_{k+1} = y_k - u[y_k] + a_{k+1}$$, which is valid

for all k and for all states y_k.
This would destroy our MGF analysis.
However, you could do a brute-force analysis for a given max number in system (N) and a given service time distribution, resulting in a particular, finite-state, DT (embedded) Markov Chain, which could be solved with $\pi = \pi P$

b) M/G/1 with infinite queue but >1 server?
Again, try to embed the MC at departure instants. But these would have to be departures from any server, and so once again it would be impossible to write a queue evolution equation similar to

$$y_{k+1} = y_k - u[y_k] + a_{k+1}$$

since in this equation, a_{k+1} is the number of arrivals in the service time of the $(k+1)^{st}$ customer. With multiple servers, it would have to be arrivals since last departure from any server, which we can't characterize.

4. With $A(1)=1$, indeterminism comes from

$$\frac{1-\frac{1}{z}}{1-A(z)/z} = \frac{z-1}{z-A(z)}$$

So use L'Hospital's rule: $\lim_{z\to 1} \frac{z-1}{z-A(z)} = \lim_{z\to 1} \frac{1}{1-A'(z)} = \frac{1}{1-\rho}$.

So we have $1=(1)\pi_0\left(\frac{1}{1-\rho}\right)$ or $\pi_0 = 1-\rho$.

5. $Y(z) = A(z)\cdot \frac{(1-\rho)(1-z)}{A(z)-z}$ $\quad A(z) =$ L.T. of service time pdf at $s=\lambda(1-z)$.

For M/M/1: $f_B(t) = \mu e^{-\mu t}$

$$A(z) = \int_0^\infty e^{-\lambda(1-z)t}\mu e^{-\mu t}\,dt = \mu \frac{1}{-(\mu+\lambda-\lambda z)}[0-1]$$

$$A(z) = \frac{\mu}{\mu+\lambda-\lambda z} \quad \text{and } A(z)-z = \frac{\mu - z(\mu+\lambda-\lambda z)}{\mu+\lambda-\lambda z}$$

Then $Y(z) = \frac{\mu(1-\rho)(1-z)}{\mu-\mu z-\lambda z+\lambda z^2} = \frac{(1-\rho)\cancel{(1-z)}}{(1-\rho z)\cancel{(1-z)}} = \frac{1-\rho}{1-\rho z}$

Taking the inverse transform:

$\pi_n = (1-\rho)\rho^n \quad n=0,1,\cdots$ as expected.

7. $\frac{1}{\mu}=2 \quad \lambda = 1/3 \Rightarrow \rho = 2/3 < 1$ (ok).

a) M/M/1: $\overline{T} = \frac{1/\mu}{1-\rho} = \frac{2}{1/3} = 6$ time units.

b) $\overline{T} = \frac{1}{\mu} + \frac{\lambda((1/\mu)^2+\sigma_B^2)}{2(1-\rho)}$ $\quad \sigma_B^2 = 0$

$$= 2 + \frac{\frac{1}{3}(2^2+0)}{2(1-2/3)} = 2 + \frac{4/3}{2/3} = 4 \text{ time units.}$$

7. c). $f_B(b) = \frac{1}{2} \quad 1 < b < 3$.

$E[B] = 2 \quad E[B^2] = \int_1^3 b^2 \left(\frac{1}{2}\right) dB = \frac{b^3}{6}\Big|_1^3 = \frac{26}{6} = \frac{13}{3}$

$$\bar{T} = \frac{1}{\mu} + \frac{\lambda E[B^2]}{2(1-\rho)} = 2 + \frac{(1/3)(13/3)}{2(1-2/3)} = 2 + 13/6 = 4.17 \text{ time units}$$

d). $f_B(b) = k(b-2)^2 \quad 1 < b < 3$.

$$\int_1^3 k(b-2)^2 db = k \frac{(b-2)^3}{3}\Big|_1^3 = k\left(\frac{1}{3} - \frac{-1}{3}\right) = k\frac{2}{3} = 1$$

$\Rightarrow k = \frac{3}{2}$ By symmetry, $E[B] = 2 = 1/\mu$.

$$\sigma_B^2 = \int_1^3 (b-2)^2 \frac{3}{2}(b-2)^2 db = \frac{3}{2}\left(\frac{1}{5}\right)(b-2)^5\Big|_1^3 = 0.3(1-(-1)) = 0.6$$

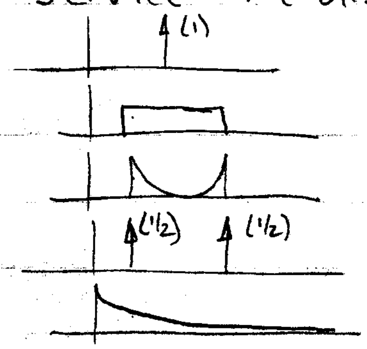
$$\bar{T} = 2 + \frac{\frac{1}{3}(2^2 + 0.6)}{2(1-2/3)} = 4.30 \text{ time units.}$$

e). $f_B(b) = \frac{1}{2}\delta(b-1) + \frac{1}{2}\delta(b-3) \quad E[B] = 2 = 1/\mu$.

$$E[B^2] = \int_{-\infty}^{\infty} \frac{1}{2} b^2 [\delta(b-1) + \delta(b-3)] db = \frac{1}{2}[1^2 + 3^2] = 5$$

$$\bar{T} = 2 + \frac{(1/3)(5)}{2(1-2/3)} = 2 + \frac{5}{2} = 4.50 \text{ time units.}$$

f). Service time dist — Mean Time in System



4 units. min variance

4.17

4.3.

4.5 Max variance given limits

6.0

6. (a) $$E(n) = \frac{dP(z)}{dz}\Big|_{z=1} = \frac{d}{dz}\left(\frac{1-\rho}{1-\rho z}\right)\Big|_{z=1}$$

$$= \frac{(1-\rho z)\cdot 0 - (1-\rho)(-\rho)}{(1-\rho z)^2}\Big|_{z=1} = \frac{(1-\rho)(\rho)}{(1-\rho)^2}$$

$$= \frac{\rho}{1-\rho} = E(n)$$

(b) $$\frac{d^2P(z)}{dz}\Big|_{z=1} = \frac{d}{dz}\frac{-(\rho^2-\rho)}{(1-\rho z)^2}\Big|_{z=1} = \frac{+(\rho^2-\rho)2(1-\rho z)(-\rho)}{(1-\rho z)^4}\Big|_{z=1}$$

$$= \frac{-2\rho^2(\rho-1)}{(1-\rho)^3} = \frac{2\rho^2(1-\rho)}{(1-\rho)^3}$$

$$= \frac{2\rho^2}{(1-\rho)^2} = E(n^2) - E(n)$$

So:

$$E(n^2) = \frac{2\rho^2}{(1-\rho)^2} + \frac{\rho}{(1-\rho)}$$

$$= \frac{2\rho^2 + \rho(1-\rho)}{(1-\rho)^2}$$

$$= \frac{2\rho^2 + \rho - \rho^2}{(1-\rho)^2} = \frac{\rho^2+\rho}{(1-\rho)^2}$$

$$\sigma_n^2 = E(n^2) - (E(n))^2 = \frac{\rho^2+\rho}{(1-\rho)^2} - \frac{\rho^2}{(1-\rho)^2} = \frac{\rho}{(1-\rho)^2}$$