

## EECS-863 Analysis of Communication Networks

Spring Semester 2008

Assignment #1 Due 29 January 2008

Reading (Hayes/Babu): Chapter 1, Sections 2.1 - 2.6 (review as necessary), Sections 2.7 and 2.8

Supplemental Reading: Section 5.3 Intro - 5.3.1 in Shanmugan/Breipohl, Sections 3A.1 - 3A.3 of Appendix A (Supplement)

1. Show that if  $\mathbf{Q}$  and  $\mathbf{P}$  are both stochastic matrices of the same dimension, then  $\mathbf{R}=\mathbf{QP}$  is also a stochastic matrix. Use this result to prove that if  $\mathbf{P}$  is a stochastic matrix, then  $\mathbf{P}^n$  is also a stochastic matrix for all natural numbers  $n$ .
2. For the 3-state ATM traffic source example given in lecture, use the transition matrix  $\mathbf{P}$  that was given in lecture:

$$\mathbf{P} = \begin{bmatrix} 0.0 & 0.4 & 0.6 \\ 0.1 & 0.2 & 0.7 \\ 0.5 & 0.0 & 0.5 \end{bmatrix}$$

Suppose the emission probabilities for the 3 states are  $e_0=0.2$ ,  $e_1=0.5$ , and  $e_2=0.8$  and the initial state pmf to be  $\boldsymbol{\pi}(0)=[0.1 \ 0.5 \ 0.4]$  (this is different from the initial state pmf used in lecture).

- a. Calculate  $\boldsymbol{\pi}(n)$  for  $n=1, 2, 3, 4, 5$ .
  - b. Calculate the probability that the source will emit a cell at time  $n$  for  $n=0, 1, 2, 3, 4, 5$ .
3. Modify Example 2.19 on p. 60 of Hayes/Babu in the following way. Let  $p_1$  be the probability that a slot of input line 1 contains a packet and  $p_2$  be the probability that a slot of input line 2 contains a packet. All else remains the same.
    - a. Draw the state transition diagram and find the transition matrix  $\mathbf{P}$ , both in terms of  $p_1$  and  $p_2$ .
    - b. If the multiplexer is full (state=2) in slot  $t$ , what is the probability that there will be one packet loss in slot  $t+1$ ?
    - c. If the multiplexer's buffer is empty (state=0) in slot  $t$ , what is the probability that it will be full (state=2) in slot  $t+2$ ?
  4. This problem will lead you through a derivation of the Chapman-Kolmogorov equations based on fundamental probability relationships (rather than matrix manipulations). To begin with, a basic relationship for joint probabilities leads to:

$$Pr[x(n) = i, x(n+m+l) = j] = \sum_k Pr[x(n) = i, x(n+m) = k, x(n+m+l) = j]$$

Working first with the left-hand side of this expression and then with the right-hand side, use basic probability rules, the Markov property and notation from Markov chains to show that this leads to the following expression for discrete-time (DT) Markov chains.

$$\pi_i(n) \cdot p_{ij}(m+l) = \sum_k \pi_i(n) \cdot p_{ik}(m) \cdot p_{kj}(l)$$

Dividing through by  $\pi_i(n)$  then yields the C-K equations.

5. Let  $\boldsymbol{\pi}$  be the solution to  $\boldsymbol{\pi} = \boldsymbol{\pi} \cdot \mathbf{P}$  for a given homogeneous DTMC with transition probability matrix  $\mathbf{P}$ . Show that if  $\boldsymbol{\pi}(0) = \boldsymbol{\pi}$ , then the DTMC is strictly stationary. That is, show that the marginal pmf  $\boldsymbol{\pi}(n)$  is independent of time ( $n$ ) and that the joint statistics are dependent only on time differences. Hint (for second part): use the result of the first part and parts of the C-K derivation from the previous problem.
6. As stated in lecture, a homogeneous DTMC is *not* irreducible if either of the following is true:

$$p_{ii} = 1 \quad \text{for some } i$$

or there exists a *strict* subset of states  $S'$  (an *absorbing* subset) such that

$$\sum_{j \in S'} p_{ij} = 1 \quad \text{for all } i \in S'$$

Note that the first condition is just a specific case of the second condition.

Express the second condition in terms of state diagrams and give an example of a DTMC that is not irreducible, showing its state diagram and the set of absorbing states.

7. Using the definition of periodicity for a homogeneous, irreducible DTMC given in lecture, show that if  $p_{ii} > 0$  for some  $i$ , then the DTMC is aperiodic.