

EECS-863 Analysis of Communication Networks

Spring Semester 2008

Assignment #3 Due 12 February 2008

Reading (Hayes/Babu): sections 2.8.3, 3.1, 3.3

Supplemental Reading: Class notes; Shanmugan/Breipohl sections 5.3.2, 5.3.3, 5.4 Intro, 5.4.1, and 5.4.2; Appendix A in the Supplement (focus on CTMCs); and Robertazzi sections 2.1 - 2.3, 2.5

1. A piece of equipment can be in one of three states: broken (state 0), under-test (state 1), and in-service (state 2). Broken equipment that has been repaired is always tested before it is put into service. If it fails the tests, it is considered broken and sent back to the shop for further repair; otherwise it is put into service. Assume its transitions between these states can be modeled by a CTMC with intensity of transition (transition rate) from in-service to broken q_{20} , from broken to under-test q_{01} , from under-test to in-service q_{12} , and from under-test to broken q_{10} .
 - a. Find the transition rate matrix and draw the state transition diagram for this system.
 - b. Write the set of differential equations that govern the evolution of the state probabilities $\pi_i(t)$.
 - c. Find the embedded (DT) Markov chain and draw its state transition diagram.
 - d. Is the original Markov chain irreducible and recurrent? If so, find the stationary state pmf $[\pi_0 \ \pi_1 \ \pi_2]$ of the original CT Markov chain.
 - e. What is the mean sojourn time in each state once the system has reached steady state?
 - f. Give an interpretation or physical explanation to each of the q_{ij} transition rates.
2. Derive a different form (called the backward form) of the differential equation relating the $P(\tau)$ matrix and the Q matrix as follows. Start with the following form of the Chapman-Kolmogorov equation:

$$p_{ij}(\tau + dt) = \sum_k p_{ik}(dt)p_{kj}(\tau)$$
 Then separate out the $k = i$ term and proceed in a similar manner as in the lecture.

3. We obtained in class one form of the global balance equations (dynamic equilibrium) for a CTMC, the form that says the total rate of transitions into a state must equal the total rate of transitions out of that state. Derive the "partition" form of the global balance equations for a CTMC, which says that if the states are partitioned into two subsets S_1 and S_2 , the total rate of transitions from S_1 to S_2 must equal the total rate of transitions from S_2 to S_1 .
4. Problem 3.5, p. 107. Simple application of Poisson process. Be careful with part (b): you will be dealing with a conditional probability (you *know* that 10 messages were generated in the first second).
5. Let A be a random variable representing the time between arrivals of taxicabs at a given location. Assume that A is distributed according to $Pr(A \leq t) = 1 - e^{-0.2t}$ for $t > 0$, where t is in minutes.
 - a. What is the average time between taxi arrivals?

- b. Suppose you arrive at the given location (on a windy, rainy night with no raincoat or umbrella) and a bystander tells you that the last cab arrived (and left) exactly 3 minutes before you arrived. How long should you expect to wait until the next taxi arrives (i.e., what is your expected waiting time for the next taxi)?
6. Consider a pure birth process with constant birth rate λ . Consider an interval of length T , which we divide up into m segments each of length T/m . Define $\Delta t = T/m$.
- a. For Δt small, find the probability that a single arrival occurs in each of exactly k of the m intervals and that no arrivals occur in the remaining $m - k$ intervals. This is a special form of what distribution?
- b. Consider the limit as $\Delta t \rightarrow 0$, that is, as $m \rightarrow \infty$ for fixed T , and evaluate the probability that exactly k arrivals occur in the interval of length T . What have you just derived?
7. Consider a population of bacteria of size $N(t)$ at time t for which $N(0) = 1$. We consider this to be a pure birth process in which any member of the population will split into two new members in the interval $(t, t + \Delta t)$ with probability approaching $\lambda \Delta t$ or will remain unchanged in this interval with probability approaching $1 - \lambda \Delta t$ as $\Delta t \rightarrow 0$.
- a. Draw the state transition diagram, labeling all transition rates, and give the transition rate matrix.
- b. Let $\pi_k(t) = Pr[N(t) = k]$ and write down the set of differential-difference equations that must be satisfied by these probabilities.
- c. Is this Markov chain recurrent or transient?
8. Consider a pure-death process with constant death rate μ that starts with an infinite initial population. How does this process differ from the pure-birth, constant birth rate (Poisson) process? How is it similar? In particular, what is the pmf of the number of deaths in a τ second interval and the pdf of the inter-death time? You need not give a formal proof; you may reason from the similarities to the Poisson arrival process.
9. We obtained a result in class that said that the interarrival time of a Poisson process (i.e., the sojourn time of a pure-birth, constant birth-rate Markov chain) has an exponential distribution. Generalize this result to show that the sojourn time in any state of any homogeneous CTMC is exponentially distributed, and find the mean of the distribution (that is, the parameter of the exponential distribution) in terms of transition rates.