EECS-863 Analysis of Communication Networks
Spring Semester 2008
Assignment **#8** Due **1 April 2008**

Reading (Hayes/Babu): sections 4.2, 6.1.1 - 6.1.3
Supplemental Reading: Class notes, Robertazzi section 2.4

1. In this problem, you will use the relationship between MGF of number in system and Laplace transform of time in system to generalize Little's Law. Recall that we used this relationship in the third derivation of Little's Law.
    a. Express the second moment of the time in the system in terms of some form of a moment of the number in the system. Keep the expression simple.
    b. Repeat for the third moment.
    c. Generalize the result for the n^{th} moment.
2. This problem will lead you to complete the proof of the statement: if the local balance equations hold for a random process $X(t)$, then the process is reversible. Assume that the local balance equations hold, then start with the situation described in lecture, in which we were considering the interval from -T to T, partitioned into intervals t_1 to t_m. Note that in the book's discussion of this topic, they use k_l instead of t_l, q_i instead of λ_i, and p_i instead of π_i.
    a. Knowing (from a previous assignment) that the sojourn time in any state is exponentially distributed, what is the pdf (probability density function) of t_l?
    b. What is the probability that state i_{l+1} will be the state following state i_l?
    c. What is the probability that the process $X(t)$ stays in state i_m for *at least* t_m time?
    d. Putting the last 3 parts together, what is a simplified expression for the joint pdf of $t_1, t_2, \cdots, t_m$?
    e. Now consider the "time-reversed" situation in which the process $X(-t)$ starts in state i_m at -T, stays for time t_m, then transitions to state i_{m-1} for time t_{m-1}, and so on, finally moving to state i_1 for *at least* time t_1. What is the joint pdf of $t_m, t_{m-1}, \cdots, t_1$?
    f. Show that the two joint pdf's are the same by invoking the local balance assumption repeatedly.
    g. What final step (assumption on the process $X(t)$) is required to complete the proof (according to the definition of reversibility)?
3. Hayes/Babu, Problem 6.1, p. 275. Trying to expand the M/G/1 model.
4. In developing the Pollaczek-Khinchine transform equation, we had an intermediate result of: $Y(z) = A(z)\,\dfrac{\pi_0(1-1/z)}{1-A(z)/z}$. Use $A(1) = Y(1) = 1$, L'Hospital's rule, and the result for $A'(1)$ to show that π_0 must equal $1-\rho$.
5. Using the Pollaczek-Khinchine transform equation for M/G/1 systems, derive the pmf for number in system for an M/M/1 system. Use the standard notation of λ for mean arrival rate, μ for mean service rate, and ρ for λ/μ.

6. This problem is taken from the Robertazzi text. Using the moment generating function (MGF) of the M/M/1 queuing system:

   a. Find the expected number in the system $E(n)$.

   b. Find the variance of the number in the system σ_n^2.

   Notice that you will have found the MGF during the previous problem.

7. This problem will use M/G/1 results to explore the issue of mean delay (time in system) for various different packet size distributions in a packet transmission system. In all cases, we will take the mean service time to be 2 time units and the mean arrival rate (Poisson) to be 1/3 per time unit.

   a. First find mean packet delay (time in system) if the packet service times are exponentially distributed.

   b. At one extreme of packet service time distributions is deterministic service time. Calculate mean packet delay (time in system) in this case.

   c. For many protocols, there are minimum and maximum packet size limits. Suppose now that the packet service times are uniformly distributed between 1 and 3 time units. Calculate mean packet delay (time in system) in this case.

   d. It was claimed in lecture that some packet length distributions (for example, Ethernet packets) have a bi-modal distribution. To model this case, assume the service times have a parabolic distribution as follows: $f_B(b) = K(b-2)^2$ for $1 < b < 3$, where K must be chosen so that this is a valid pdf. Calculate mean packet delay (time in system) in this case.

   e. An extreme bimodal distribution would be discrete with equal 50% probability for service times of either 1 or 3 time units. Calculate mean packet delay (time in system) in this case.

   f. Summarize all of your results for this problem in a table, and discuss any trends that you see.