HW #2 Solutions

Corollary p185.

1. Pf: Let x^* be a relative minimum pt of f over Ω and an interior pt. of Ω. Let d be any arbitrary direction (all are feasible). Then Prop 1 says that
$$\nabla f(x^*)d \geq 0.$$
Since every direction is feasible, $-d$ is, so again by Prop 1 we have
$$\nabla f(x^*)[-d] = -\nabla f(x^*)d \geq 0 \Rightarrow \nabla f(x^*)d \leq 0.$$
Putting together the two inequalities, we have
$$\nabla f(x^*)d = 0.$$
Since d was arbitrary, this must hold for all d, which implies $\nabla f(x^*) = 0$.

Linear Prediction

2. a) $f(a) = E\left[\left(X_k - (a_1 X_{k-1} + a_2 X_{k-2} + \cdots + a_p X_{k-p})\right)^2\right]$.

FONC: $\nabla f(a) = 0$, i.e. $\frac{\partial f(a)}{\partial a_i} = 0 \ \forall i$.

Note first that $\frac{\partial}{\partial a_i} E[g(a)] = E\left[\frac{\partial}{\partial a_i} g(a)\right]$. Then -

$$\frac{\partial}{\partial a_i} f(a) = E\left[2\left(X_k - \sum_{j=1}^{p} a_j X_{k-j}\right) X_{k-i}\right]$$

$$= 2E[X_k X_{k-i}] - 2\sum_{j=1}^{p} a_j E[X_{k-j} X_{k-i}] = 0$$

$$R_X(i) = \sum_{j=1}^{p} a_j R_X(i-j)$$

where $R_X(l)$ is the lag-l autocorrelation of the stationary process $\{X_k\}$.

b). For $p=1$, this becomes $a_1 = \frac{R_X(1)}{R_X(0)}$ which is the normalized lag-1 autocorrelation

c). The power spectral density (PSD) of the filter output has the form $\frac{C_1}{\left(\frac{1}{RC}\right)^2 + (2\pi f)^2}$, so the autocorrelation of the filter output is of the form $C_2 e^{-\frac{|\tau|}{RC}}$. Then

$$R_X(l) = C_2 e^{-\left|\frac{l}{8000RC}\right|} = C_2 e^{-|l/8|}$$

2c) (cont) The FONC for $p=3$ are then the equations:

$$e^{-1/8} = a_1 e^{0} + a_2 e^{-1/8} + a_3 e^{-2/8}$$
$$e^{-2/8} = a_1 e^{-1/8} + a_2 e^{0} + a_3 e^{-1/8}$$
$$e^{-3/8} = a_1 e^{-2/8} + a_2 e^{-1/8} + a_3 e^{0}$$

or
$$\begin{bmatrix} 1 & e^{-1/8} & e^{-2/8} \\ e^{-1/8} & 1 & e^{-1/8} \\ e^{-2/8} & e^{-1/8} & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} e^{-1/8} \\ e^{-2/8} \\ e^{-3/8} \end{bmatrix}$$

and the solution is $a_1 = e^{-1/8} = .882$ $a_2 = a_3 = 0$. It is easy to see that this surprising result holds for any RC, any sampling frequency f_s and any p, that is, $a_1 = e^{-\frac{1}{f_s RC}}$ $a_2 = \cdots = a_p = 0$. This is one reason for calling an RC lowpass filter a "first order" filter -- in digital form, it would appear as an FIR filter with its only non-zero coefficient at lag 1.

d). $E[e_k^2] = E[(x_k - e^{-1/8} x_{k-1})^2] = R_x(0) - 2e^{-1/8} R_x(1) + e^{-2/8} R_x(...$

$$\Rightarrow 1/G_p = E[e_k^2]/R_x(0) = 1 - 2e^{-1/8}(e^{-1/8}) + e^{-2/8} = 1 - e^{-1/4} = 0.22$$
$$\Rightarrow G_p = 4.5 \quad (6.5\text{ dB}).$$

3. Simple matrix algebra (though a little tedious).

4. $f_1(x) = a^T x = \sum_i a_i x_i$ x and a are <u>column</u> vectors.

$\nabla f_1(x) = a^T$ $\quad F_1(x) = 0$ ∇f is a <u>row</u> vector

$f_2(x) = x^T A x$ with A symmetric.

$$f_2(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} x_j$$

$$\frac{\partial f_2(x)}{\partial x_k} = \sum_{j \neq k} a_{kj} x_j + 2 a_{kk} x_x + \sum_{i \neq k} a_{ik} x_i$$

$$= \sum_j a_{kj} x_j + \sum_i x_i a_{ik}$$

$$= (k^{th} \text{ row of } A) \cdot x + x^T (k^{th} \text{ col of } A)$$

$$= 2 x^T \cdot (k^{th} \text{ col of } A) \quad \text{since } A \text{ is symmetric.}$$

So, $\nabla f_2(x) = 2 x^T A$ (a row vector).

Continuing, since $\dfrac{\partial f_2(x)}{\partial x_k} = 2 \sum_i x_i a_{ik} = 2 \sum_j a_{kj} x_j$

Then $\dfrac{\partial f_2(x)}{\partial x_k \partial x_\ell} = 2 a_{k\ell} \Rightarrow F_2(x) = 2A$

5. $f_1(x) = 2x_1^2 + x_2^2 - 3x_1 - x_1x_2$

$\nabla f_1(x) = (4x_1 - x_2 - 3\ ,\ -x_1 + 2x_2)$

FONC for interior points (all points interior here):

$\nabla f_1(x) = (0\ ,\ 0) \Rightarrow \quad 4x_1 - x_2 - 3 = 0$

$\qquad\qquad\qquad\qquad\quad -x_1 + 2x_2 = 0$

Solving those 2 simultaneous linear eqns:

$x_1 = 6/7$ and $x_2 = 3/7$ satisfies FONC.

Check 2nd order conditions:

$\mathbb{F}_1 = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$ Test for positive-definite by checking determinant of each principal sub-matrix

$\left.\begin{array}{l} \det(4) = 4 > 0 \\ \det(\mathbb{F}_1) = 7 > 0 \end{array}\right\}$ $\mathbb{F}_1$ is positive-definite,

so the point $(6/7, 3/7)^T$ <u>is</u> a rel. min. pt. of $f_1(x)$.

$f_2(x) = 2x_2^3 + 3x_1^2 + 6x_1x_2 + 12x_1$ No constraints.

$\nabla f_2(x) = (6x_1 + 6x_2 + 12\ ,\ 6x_1 + 6x_2^2)$

<u>FONC</u>: $6x_1 + 6x_2 + 12 = 0$ <u>and</u> $6x_1 + 6x_2^2 = 0$.

Fortunately, the 1st eqn is linear, so $x_1 = -x_2 - 2$.

Now $6x_1 + 6x_2^2 = 0 \Rightarrow x_1 + x_2^2 = 0 \Rightarrow -x_2 - 2 + x_2^2 = 0$.

Using quadratic formula, $x_2 = 2$ or $x_2 = -1$.

For $x_2 = 2$, $x_1 = -4$ and for $x_2 = -1$, $x_1 = -1$.

So, two points meet FONC: $\underbrace{(-4,2)^T \text{ and } (-1,-1)^T}_{\text{These are possible rel. min. pts.}}$.

Check 2nd order conditions:

$\mathbb{F}_2 = \begin{pmatrix} 6 & 6 \\ 6 & 12x_2 \end{pmatrix}$ $\det(6) = 6$, $\det(\mathbb{F}_2) = 72x_2 - 36$

$(-4,2)^T$: $\det \mathbb{F}_2 = 144 - 36 > 0 \Rightarrow$ meets SOSC and <u>is</u> rel min. pt.

$(-1,-1)^T$: $\det \mathbb{F}_2 = -72 - 36 < 0 \Rightarrow$ fails SONC and is <u>not</u> a rel min pt.

6. $\max f(x,y,z) = \dfrac{xyz}{x+2y+2z^2}$ s.t. $xy=2$ and $x,y,z \geq 0$.
substitute: $x = 2/y$ This is ok since $y \geq 0$ implies $x \geq 0$.

Now we have $\max g(y,z) = \dfrac{2z}{2/y + 2y + 2z^2} = \dfrac{z}{1/y + y + z^2}$ s.t. $y,z \geq 0$
Note that $g(y,z) \geq 0$ for all y,z.
Since $a \geq b \iff \frac{1}{a} \leq \frac{1}{b}$ for all $a \geq 0$ and $b \geq 0$, maximizing g is equivalent to minimizing $\frac{1}{g}$. Letting $h = \frac{1}{g}$, we have:

$\min h(y,z) = \dfrac{\frac{1}{y} + y + z^2}{z}$ s.t. $y,z \geq 0$.
FONC for interior points: $\nabla h = 0$

$$0 = \frac{\partial h}{\partial y} = \frac{z(-y^{-2}+1)-0}{z^2} = \frac{-\frac{1}{y^2}+1}{z} \Rightarrow y^2 = 1 \Rightarrow \underset{\text{ok}}{y=1} \text{ or } \underset{\text{not allowed}}{y=-1}.$$

$$0 = \frac{\partial h}{\partial z} = \frac{z(2z)-(y^{-1}+y+z^2)}{z^2} = \frac{z^2 - \frac{1}{y} - y}{z^2} \text{ with } y=1,\ z^2=2$$
$$\underset{\text{ok}}{z=\sqrt{2}} \text{ or } \underset{\text{not allowed}}{z=-\sqrt{2}}.$$

So, $y=1$ $z=\sqrt{2}$ satisfies FONC and may be a rel. min. pt. Since it is an interior pt, check 2nd order conditions:

$$H = \begin{pmatrix} \frac{2y^{-3}}{z} & \frac{y^{-2}-1}{z^2} \\ \frac{y^{-2}-1}{z^2} & \frac{y^{-1}+y}{z^4} \end{pmatrix} \qquad H(1,\sqrt{2}) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Check for $H(1,\sqrt{2})$ pos definite:
$\det(\sqrt{2}) = \sqrt{2} > 0 \quad \det(H) = \frac{\sqrt{2}}{2} > 0 \Rightarrow H$ pos. def.
So $y=1$ $z=\sqrt{2}$ is rel. min. pt. of h s.t. $y,z \geq 0$ and so $(2,1,\sqrt{2})^T$ is a rel max point of f s.t. $x,y,z \geq 0$.
The value of f at this point is 0.3536. For any point on boundary ($x=0$ or $y=0$ or $z=0$) $f=0$, so $(2,1,\sqrt{2})$ is global max of f s.t. $x,y,z \geq 0$.