The University of Kansas

Information and
Telecommunication
Technology Center

Technical Report

Differential Pricing for Differentiated Services

Yuhong Liu and David W. Petr

ITTC-FY2001-TR-18836-05

August 2000

Project Sponsor:
Sprint Corporation

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1 Introduction

Currently the Internet only offers a single type of service: all packets are serviced on a best-effort, first-in-first-out (FIFO) basis. This single type of service limits the nature of applications that can be adequately supported. As the "information revolution" is currently underway, more and more new applications are emerging, and the Internet has to support a wide variety of applications that have very different service requirements. For instance, some applications, like electronic mail, can tolerate significant delay without users experiencing discernible performance degradation, while other applications, such as voice over IP, degrade perceptibly with even extremely small delays. Similarly, some applications are relatively insensitive to packet loss while others are not, and some applications can adjust to reduced bandwidth while others cannot. The range of applications, and the diversity of service requirements, is likely to grow rapidly in the near future. Thus, it is crucial for the evolution of the Internet to meet these increasingly varied service requirements, thus becoming an integrated network.

An integrated network caters to diverse application requirements, and an efficient solution to the problem of providing adequate quality of service (QoS) to heterogeneous users is through the offer of multiple service classes. With the emerging protocols such as RSVP (Resource Reservation Setup Protocol) [5], DIFFSERV [6] and scheduling schemes such as priority queuing, class-based queuing and Weighted Fair Queuing (WFQ), the Internet is also going to be able to provide a collection of various classes of service.

However, the current pricing and charging methods for Internet service are mainly flat-rate access-based, which will not be sufficient in the presence of multiple service classes and discrimination between different usage requests [3].

Recently several papers have shown that differentiated pricing could provide incentives to users to improve network utilization, network efficiency and fairness among the users. Gupta *et al* [1] propose a priority pricing scheme to manage a multi-service class network which can be implemented in a completely decentralized environment. They show through simulation that priority pricing improves the performance significantly as compared to a uniform pricing scheme.

Cocci *et al* [2][3] study the role of pricing policies in multiple service class networks (specifically, non-preemptive priority network). They conclude via simulations that with the same revenue generated, the users are more satisfied with the graduated pricing scheme than the uniform pricing scheme.

DaSilva *et al* [4] adopt the game theory approach to analyze the existence and uniqueness of the Nash equilibrium for a non-cooperative "game". The authors argue that by appropriately selecting the different prices for various priority levels, the network providers will be able to induce an optimal point that can maximize both revenue and aggregated utility.

In this technical report, we study the revenues raised by different pricing schemes in an analytical framework. As with some previous studies [3][4], we adopt a game theory approach to address this problem. We model the system as a game, and all the users are the participants of this game. The service provider sets the unit prices to maximize the total revenue, and all the users request the service to optimize their individual levels of satisfaction. Game theory has been widely used in economics to address similar problems. Here we consider two different pricing schemes. The first one is a uniform price scheme applies a uniform unit price to all the packets traversing a link. The second one is a differential pricing scheme in which each priority class has a different unit price. We compare the revenues raised by the differential pricing scheme and the uniform pricing scheme and show that if the users' delay requirements are sufficiently different, the differential pricing scheme will raise more revenue for the service provider without losing any potential customers.

This technical report is organized as follows: section 2 is an introduction to game theory. Sections 3 and 4 introduce separately the network model and the user model. Section 5 explains how to get the optimal unit prices and the optimal revenues. Section 6 compares the optimal revenues raised by the two pricing schemes. Finally section 7 addresses some related further research fields. The Appendix contains the derivation of the optimal differential unit prices.

2 Game theory

As defined in [4], a game consists of a principal and a finite set of players. Each player will choose a strategy with the objective of maximizing his surplus function. If the users make choices independently, this is a non-cooperative game. Game theory attempts to predict the outcome of such a game, or properties of the predicted outcome. A particularly satisfying outcome of a non-cooperate game has the property that no user, by unilaterally changing her own request, can increase her surplus. This is referred to as the Nash equilibrium. The Nash equilibrium is considered a consistent prediction of the outcome of the game, in the sense that if all players predict that a Nash equilibrium will occur, then no player has an incentive to choose a different strategy.

In a multi-service network, the service provider sets the unit prices and the service principle; each user is selfish and acts alone to make service choices with the aim of maximizing his individual level of satisfaction, *i.e.*., the surplus function. The surplus is a function of the performance of the selected service, which is in turn affected by the others' choices. Therefore, all the users' choices are interdependent. This could be modeled as a non-cooperative game, and the users are the players of the game.

The service provider is interested in determining which pricing schemes are more likely to achieve certain goals, e.g. maximizing revenue or social welfare. It is therefore important to be able to predict what service choices the customers will make under a given pricing scheme. Since the Nash equilibrium is the likely outcome of the game, it can be considered the network's operating point. The selected pricing scheme should optimize the objective function while inducing a Nash equilibrium.

According the game theory, a Nash equilibrium point is induced when unilateral deviation does not help any user improve his performance. In other words, at Nash equilibrium every user's surplus function is no less than that when he changes his choice while all the others' choices remain unchanged.

3 The network model

Here we consider a simple network with a fixed population of *N* potential network users. For analytical tractability, we simplify it to be a single trunk network. The objective of the service provider is to maximize the revenue while trying to serve all the potential customers.

3.1 Service discipline

We assume the network supports two service classes, high and low priority, by extend the first-in-first-out (FIFO) service discipline to two classes. The end node of the trunk then keeps a queue with the high priority packets arranged in the order of time-of-arrival, followed by the low priority packets also arranged in the order of time-of-arrival. The node transmits the packet at the head of the queue.

3.2 Pricing scheme

We adopt a usage-based pricing scheme here. We assume the charge for the service is a function of the number of packets served. Note that there also might be some fixed charges, like a monthly fee or connection fee, but they would be the same for the two pricing schemes. Since we are just interested in comparing these two pricing schemes, we ignore these fixed charges.

So the charge for customer i is:

$P_i = p_{c(i)} \cdot$ average number of packets served in time T

Where: $c(i)$ is the service choice made by customer i;

$p_{c(i)}$ is the price per packet of the service class chosen by customer i ,

T is the billing interval.

Under the uniform pricing approach, there is only one value of p.

In the differential pricing approach, we denote by p_1 the unit price for the high priority class, and p_2 the unit price for the low priority class.

Equivalently, since we will be comparing the two pricing strategies for identical billing intervals, we let:

$P_i = p_{c(i)} \cdot \lambda_i$

Where λ_i is the arrival rate of user *i* packets.

4 The user model

We assume each user is selfish and acts alone. A user's only objective is to request a service maximizing his individual level of satisfaction. Furthermore we assume the priority is assigned on per-user basis.

4.1 Traffic

We assume all the users have the identical traffic statistics. Furthermore, we assume that the user's traffic is independent of the prices.

We assume that each user's traffic is a Poisson process, with arriving rate λ. The average service time for each packet is x, and $\overline{x^2}$ is the second moment of the average service time.

4.2 Surplus function

The surplus function represents the level of a user's satisfaction with the service. We denote by C_i user i's surplus function, which is the difference between the utility function and the charges for the service.

The utility function U_i is how much money user i will be willing to pay for a particular service. U_i is a function of user's traffic amount and quality of service (QoS), that is, the utility function describes how sensitive a user is to changes in QoS. The unit of U_i is some monetary unit (for example, U.S dollars).

Here we choose mean delay time as the indication of the QoS, so we assume the utility is a function of the waiting time:

$$U_i = \lambda(A - B_i \cdot W_i)$$

Where: A is the upper bound of the amount of money the user is willing to pay for the service;

W_i is the waiting time experienced by user i;

B_i is a coefficient reflecting the effect of the delay time on user i's benefit function.

Note that, for user i the value of W_i when U_i goes down to 0 is his maximum tolerable waiting time.

Figure 1 shows two example utility functions with the same λ but different B_i values. This represents two users who have the same traffic but different QoS.

Figure 2 shows two example utility functions of this form with $\lambda_1 > \lambda_2$ but the same value for B_i. This represents how the utility of a given customer changes as his traffic changes.

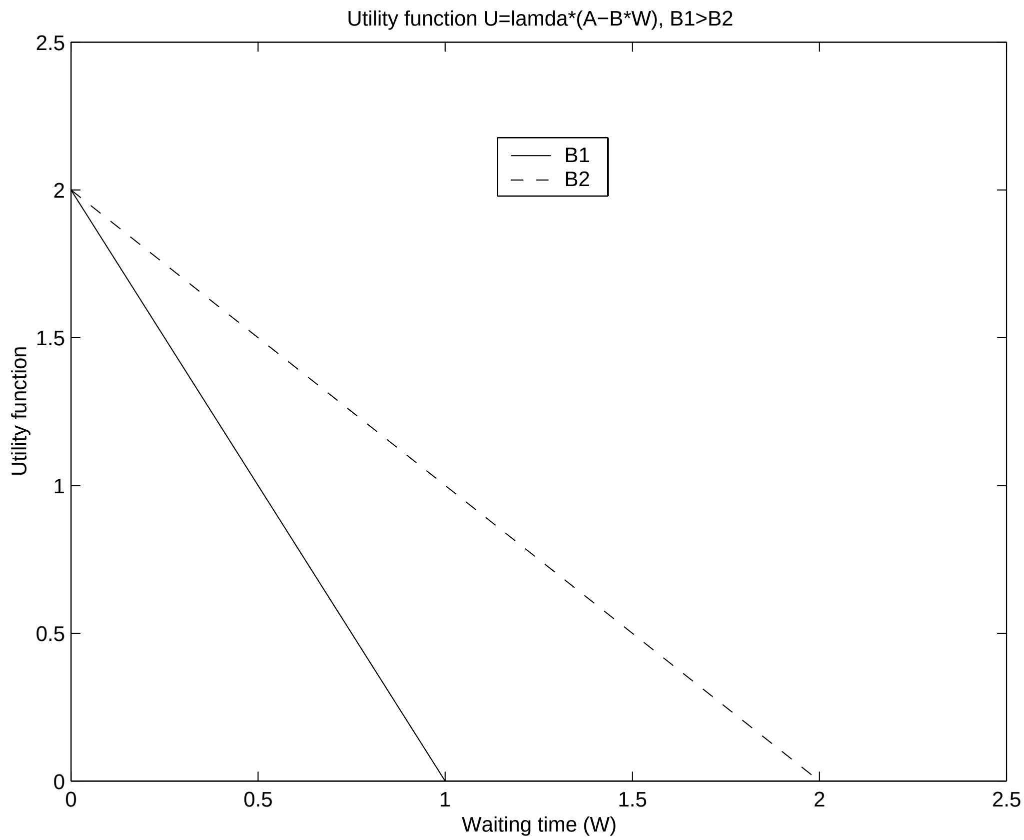

Figure 1 User's utility function

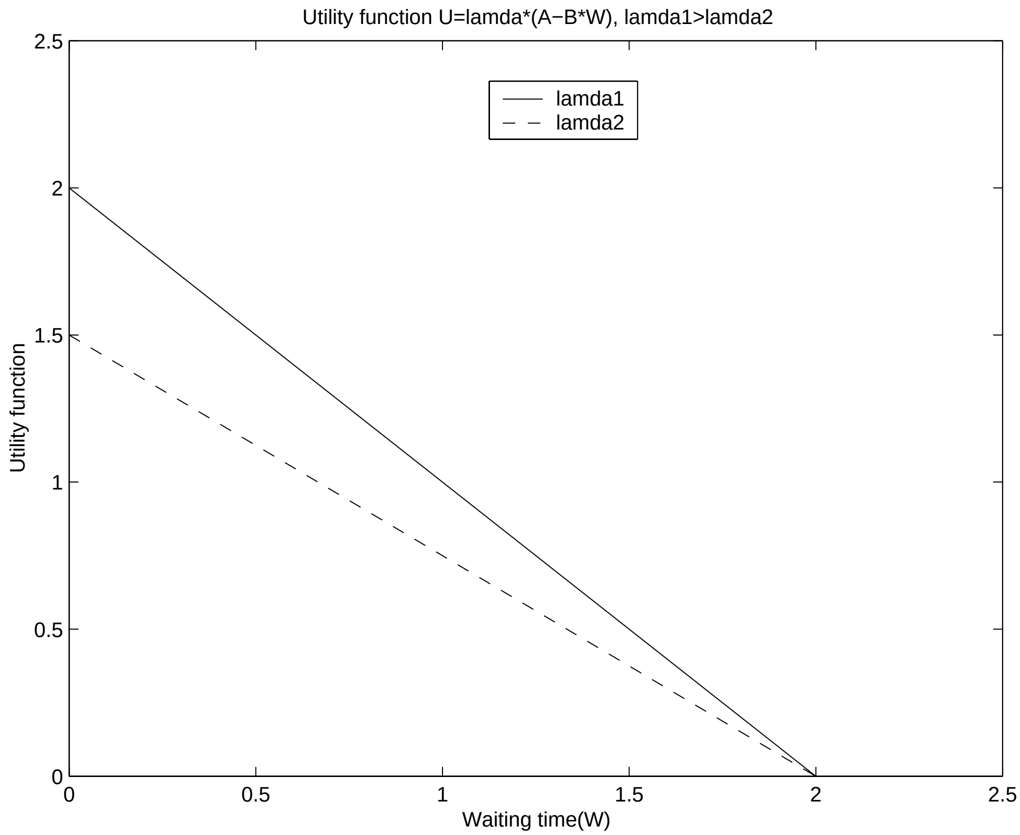

Figure 2 User's utility function

The surplus function is:

$C_i = U_i – P_i$

To a user, if the surplus function of one service goes below zero, the charges of the service exceed the expected benefit. We assume that this user will not choose this service class. Furthermore, if the surplus functions of all the service classes are below zero, the user will not choose any of the services.

Our objective now is to find the optimal unit prices p (uniform pricing) and p_1 and p_2 (differential pricing) to optimize the revenue while ensuring that each user's surplus is non-negative.

5 Optimal Revenue

This section builds on the work of [4], generalizing it to an arbitrary number of users and introducing provider revenue optimization.

5.1 Optimal Revenue Under the Uniform Pricing Scheme

Under the uniform pricing scheme, all users would choose to request high priority service. This is because there is no monetary incentive to request low priority service and, if there is any congestion in the network, there is a performance dis-incentive to request low priority service. Therefore, this situation devolves to one in which there is only a single service class, and the waiting time experienced by each user is identical. In order to serve all the N users, the service provider should set the unit price p so that each user's surplus is no less than 0.

The optimal problem is the following:

Maximize: $\sum_{i}^{N} p \cdot \lambda = N \cdot p \cdot \lambda$

Subject to: $(A - B_i \cdot W)\, \lambda - p \cdot \lambda \geq 0, \ \forall i = 1,2,\ldots,N$

Since the objective function is linear, the optimal p should take the maximum value as long as it satisfies the constraints. The constraints could further be reduced to one, which is for the largest value of B_i. So the optimal unit price is:

$$p = A - B_{\max} \cdot W$$

where: B_{max} is the maximum value of the B_i.

The optimal revenue is then:

$$N \cdot (A - B_{\max} \cdot W) \cdot \lambda$$

5.2 Optimal Revenue under the Differential Pricing Scheme

Under the differential pricing scheme, the service provider charges different unit prices for the two priority classes. Let us denote by p_1 the unit price for high priority class, and p_2 for the low priority class, with $p_1 > p_2$.

We denote by N_1 the number of the users choosing the high priority class at an operating point, N_2 as the number of the users choosing low priority, with $N_1+N_2=N$.

If $N_1 =N$, the optimal unit prices are the same as that of uniform pricing scheme. Therefore, we consider the case when $N_1 <N$.

First, we will find the optimal unit prices that maximize the revenue for every value of N_1 from 1 to N-1. Then if a service provider aims only to maximize the revenue, he can find the optimal N_1 through searching the revenues from N_1=1 to N.

In [5], Kleinrock presented the closed-form results for the mean waiting time in a two-priority head-of-line priority discipline for a single M/G/1 queue. According his results, the waiting time of high and low priority classes are as following:

$$W_1 = \frac{W_0}{(1 - N_1 \lambda x)}$$

$$W_2 = \frac{W_0}{(1 - N_1 \lambda x)(1 - N \lambda x)}$$

where:

$$W_0 = \tfrac{1}{2}(N_1 \lambda \overline{x^2} + N_2 \lambda \overline{x^2}) = \tfrac{1}{2}(N \lambda \overline{x^2})$$

Our goal now is to find the unit prices p_1 and p_2 that optimize the revenue. Given the assumption that the selfish behavior of individual users will result in a Nash equilibrium, the optimal unit prices should induce a Nash equilibrium.

The Nash equilibrium represents the point where “unilateral deviation does not help any user improve his performance.” In our model, this could be interpreted as follows: at the Nash equilibrium, the user’s surplus is no less than that if he alone shifts to another service class. In other words, the surplus function of a user in high priority class will be no less than the surplus function when he changes to low priority and all the others remain unchanged, and similarly for a user charge from low to high priorities.

Let us denote by $W_{1,+i}$ the waiting time of the high priority class when user i changes his choice from low priority to high priority and all the others remain unchanged.

$$W_{1,+i} = \frac{W_0}{[1-(N_1+1)\lambda x]}$$

Similarly, $W_{2,+j}$ is the average waiting time of the low priority class when user j alone changes his choice from high priority to low priority.

$$W_{2,+j} = \frac{W_0}{[1-(N_1-1)\lambda x](1-N\lambda x)}$$

Using this notation, to induce a Nash equilibrium where N_1 users will request high priority class and N_2 users request low priority service, the unit prices should satisfy the following condition:

- For the user in high priority class,

$$\lambda(A-B_iW_1)-p_1\lambda \geq \lambda(A-B_iW_{2,+i})-p_2\lambda\,, \quad i=1,\ldots N_1$$

$$i.e.,\ (p_1-p_2) \leq B_i(W_{2,+i}-W_1)\,, \quad \forall\ i=1,\ldots N_1$$

- For the user in low priority class,

$$\lambda(A-B_jW_2)-p_2\lambda \geq \lambda(A-B_jW_{1,+j})-p_1\lambda\,,\ j=1,\ldots N_2$$

$$i.e.,\ (p_1-p_2) \geq B_j(W_2-W_{1,+j})\,,\ \forall\ j=1,\ldots N_2$$

For the service provider, a desired operating point (Nash equilibrium) may be influenced by some other considerations. For example, if it is desired to provision an upper bound on the delay time for high priority class and the capacity is fixed, the number of users requesting high priority service should be limited to reach this upper bound. In our case, we assume the service provider simply wishes to maximize revenue.

Furthermore, according to our assumption, the service provider will try to serve all the potential users. Thus under the optimal unit prices, a user's surplus function must not be negative. That is:

$$p_1 \le A - B_i W_1, \forall i = 1,2 \ldots N_1$$

$$p_2 \le A - B_j W_2, \forall j = 1,2 \ldots N_2$$

We use the following notation:

B_{1max} is the maximum value of B_i among the users choosing high priority class,

B_{1min} is the minimum value of B_i among the users choosing high priority class

B_{2max} is the maximum value of B_j among the users choosing low priority class.

We can show (see Appendix) that B_{1min}, and hence also B_{1max} is no less than B_{2max}. Furthermore, at an equilibrium point, the users with higher value of B will choose high priority, and the users with lower values of B will choose low priority.

Using this notation, the optimization problem is:

Maximize: $\sum_{i=1}^{N_1} p_1 \lambda + \sum_{j=1}^{N_2} p_2 \lambda = N_1 p_1 \lambda + N_2 p_2 \lambda$

Subject to: $p_1 \le A - B_{1\max} W_1$

$$p_2 \le A - B_{2\max} W_2$$

$$(p_1 - p_2) \le B_{1\min}(W_{2,+i} - W_1)$$

$$(p_1 - p_2) \ge B_{2\max}(W_2 - W_{1,+j})$$

The first two constraints ensure non-negative surplus for *every* user, and the second two ensure a Nash equilibrium.

Hereafter, we adopt the following notations:

p_{1max} = A - B_{1max} W_1,

p_{2max} = A – B_{2max} W_2,

$(p_1 - p_2)_{max}$ = B_{1min} ($W_{2,+i}$ - W_1),

$(p_1 - p_2)_{min}$ = B_{2max} (W_2 - $W_{1,+j}$).

As shown in the Appendix, the optimal unit prices $p_{1optimal}$ and $p_{2optimal}$ are as follows:

Case 1: if $(p_1 - p_2)_{min} \le p_{1max} - p_{2max} \le (p_1 - p_2)_{max}$, $p_{1optimal} = p_{1max}$ and $p_{2optimal} = p_{2max}$.

Case 2: if $p_{1max} - p_{2max} < (p_1 - p_2)_{min}$, $p_{1optimal} = p_{1max}$ and

$p_{2optimal} = p_{1max} - (p_1 - p_2)_{min}$

Case 3: if $p_{1max} - p_{2max} > (p_1 - p_2)_{max}$, $p_{1optimal} = p_{2max} + (p_1 - p_2)_{max}$ and

$p_{2optimal} = p_{2max}$

Comparing the revenues of the two pricing schemes, we have the following results for a equilibrium point with N_1 users choosing high priority class:

- If N_1=N, the two pricing schemes are the same;

- If $N_1<N$, then (see Appendix for derivation):

$$\text{If} \quad B_{2\max} < B_{1\max}(1-N_1\lambda x) + B_{1\min}\{N_1 \lambda x - \frac{N_1\lambda x}{N[1-(N_1-1)\lambda x]}\},$$

the revenue raised by the differential pricing scheme is greater than the uniform pricing scheme. Otherwise, the revenue raised by the uniform pricing scheme is greater than the differential pricing scheme.

6 Discussion and Numerical Results

In this technical report we have studied a simple model, one that involves only a single node, two service classes and where the users' traffic patterns are identical. However we believe this network context is realistic since in this model the pricing issues are not obscured by the technical details of the network.

In our model, the value of B_i represents a user's sensitivity to the delay (waiting) time. Different values of B could be interpreted as the reflection of the users' different delay requirements. The results show that if the users' delay requirements are sufficiently different, the differential pricing scheme will raise more money for the service provider than the uniform pricing scheme. On the other hand, if the users' service quality requirements are similar, the uniform pricing scheme is more profitable. Given the great diversity of the quality requirements that exists in the Internet now, it is more likely that the service provider would benefit more from the differential pricing scheme.

The choice of the operating point should take into account the specific requirements for services in a network. For example, if the network is over-provisioned, and there is no need to restrict the number of users at high priority, the network provider could choose the value of N_1 close to the total users number N. If the operating point is $N_1 = N$, the two pricing schemes are the same. Otherwise, the network provider could restrict the number of users by setting the operating point at $N_1 \ll N$. In this case, a service provider could offer a delay bound for higher class to meet the requirements of some quality-sensitive applications. The preference for the two pricing schemes depends on a user's delay requirement: the users with higher quality requirements will choose high priority, and those with lower requirements will choose low priority class.

If a service provider aims only to maximize the revenue, he has to find the optimal value of N_1 that maximizes revenue. This optimal N_1 lies somewhere between 1 and N. Since the price for high priority is larger than low priority price, at the beginning the revenue increases as N_1 increases. But as N_1 increases, the average delay time of high priority class also increases and the optimal price for high priority decreases. So after the revenue reaches a maximum point, it goes down as N_1 increases. Service providers could thus find optimal N_1 through searching the revenues from N_1=1 to N.

Figure 3 shows some revenue curves as a function of N_1 under different users' service requirements for a network. In this case, the network has 5 potential users and all of them have the same traffic characteristics. We let each user's traffic arriving rate to be 1, and set A to be 28. We also let $W_0 = 0.05 \cdot \lambda$. We can see that if the user population has service requirements that are differentiated enough (B=2.5, 10, 50, 100, 250) revenues from the differential pricing scheme exceed uniform pricing scheme for all N_1 from 1 to $N-1$. Note that the revenue at $N_1=N$ is the same as the uniform pricing scheme. Here we choose the values of B_i such that they represent users' maximum tolerable waiting time range from 0.1s (for real time applications) to 10s (for email applications). As user's service requirements become more and more similar, at some values of N_1 the revenues will be less than the uniform pricing scheme. If users' service requirements are very close (B=230, 235, 245, 250, which means most of users have the maximum tolerable waiting time at the same level as voice), the uniform pricing scheme will become more beneficial.

Figures 4 and 5 demonstrate the effect of users' traffic utilization or these two figures, the parameters are the same. Figure 4 shows the difference between the two pricing schemes becomes greater when user's traffic utilization increases. Here we choose B=2.5, 10, 50, 100, 250, and let $W_0 = \frac{1}{2}(N\lambda\overline{x^2}) = 0.05 \cdot \lambda$, and set A to be 28. We can see the two pricing scheme are close to each other while λ=0.8, and the difference increases for λ=1. Note that the revenue under the lower traffic utilization is greater than the higher utilization. This is because the willingness-to-pay (user's utility function) decreases as the average waiting times increases. Figure 5 further shows that the difference between the optimal revenues from the two pricing schemes increases as user's traffic utilization increases. The parameters for this figure are: N=5, x=0.1, A=28, $W_0 = \frac{1}{2}(N\lambda \overline{x^2}) = 0.05 \cdot \lambda$, B=2.5, 10, 50, 100, 250. Here the optimal revenues under differential pricing scheme are the maximum point among N_1=1,2,…N for a specific value of λ. This shows that the differential pricing scheme is more beneficial for a network with higher utilization. Note that when link utilization is 0.5, the revenue from uniform pricing is just a little above 0, which means it is almost non-profitable for a service provider although the utilization is not particularly high. On the other hand, differential pricing scheme can help the service provider to make profit without losing users at this point.

Furthermore, our results could be viewed as a comparison of a priority network with a best-effort network. The revenue of uniform pricing scheme is the same as a best-effort network without differential service classes. The results show that if users' service requirements are sufficiently different, provisioning multi-service classes will generate more revenue for a service provider.

7 Summary of lessons learned

- Differential pricing is superior to uniform pricing if the user's delay sensitivities are sufficiently different
- Advantage of differential pricing can be significant for a large spread in users' delay sensitivities
- Advantage of differential pricing increases with increasing traffic
- Differential pricing allows profitable operation at higher loads

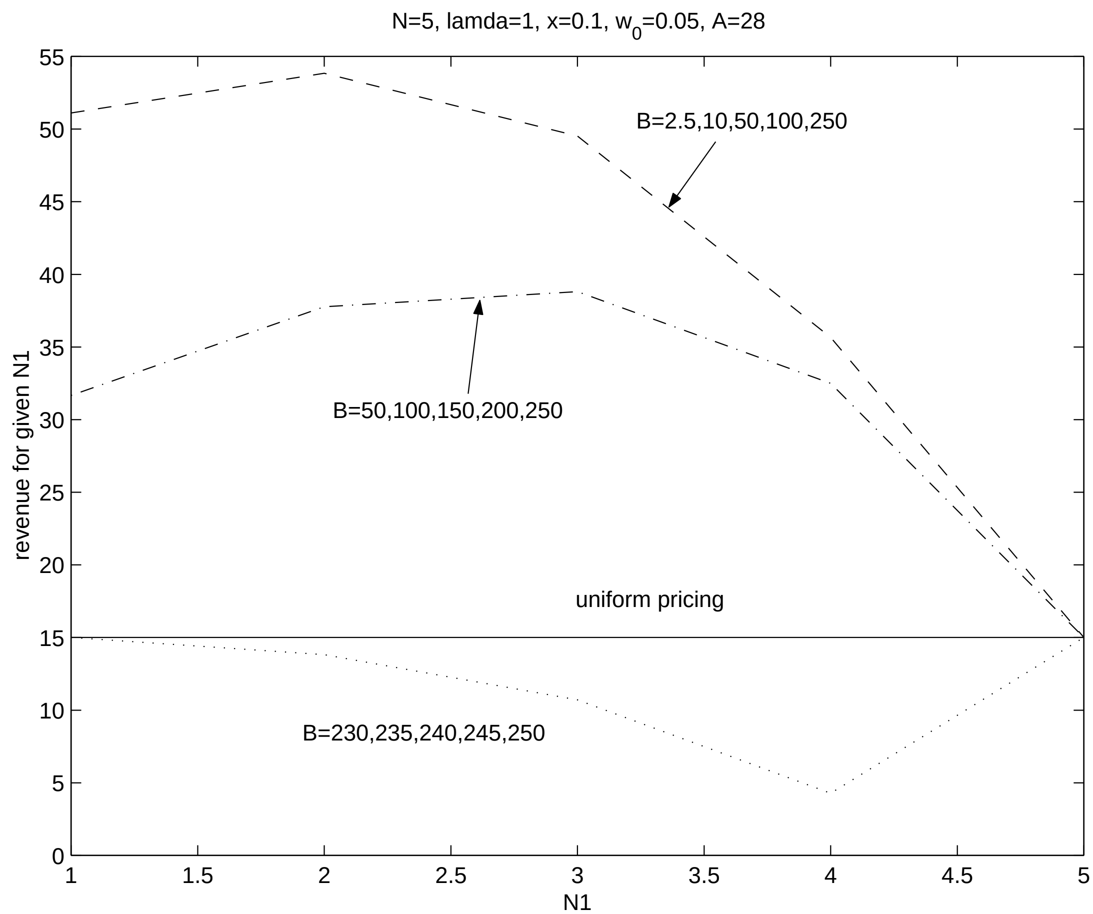

Figure 3 Optimal revenues under different users' service requirements

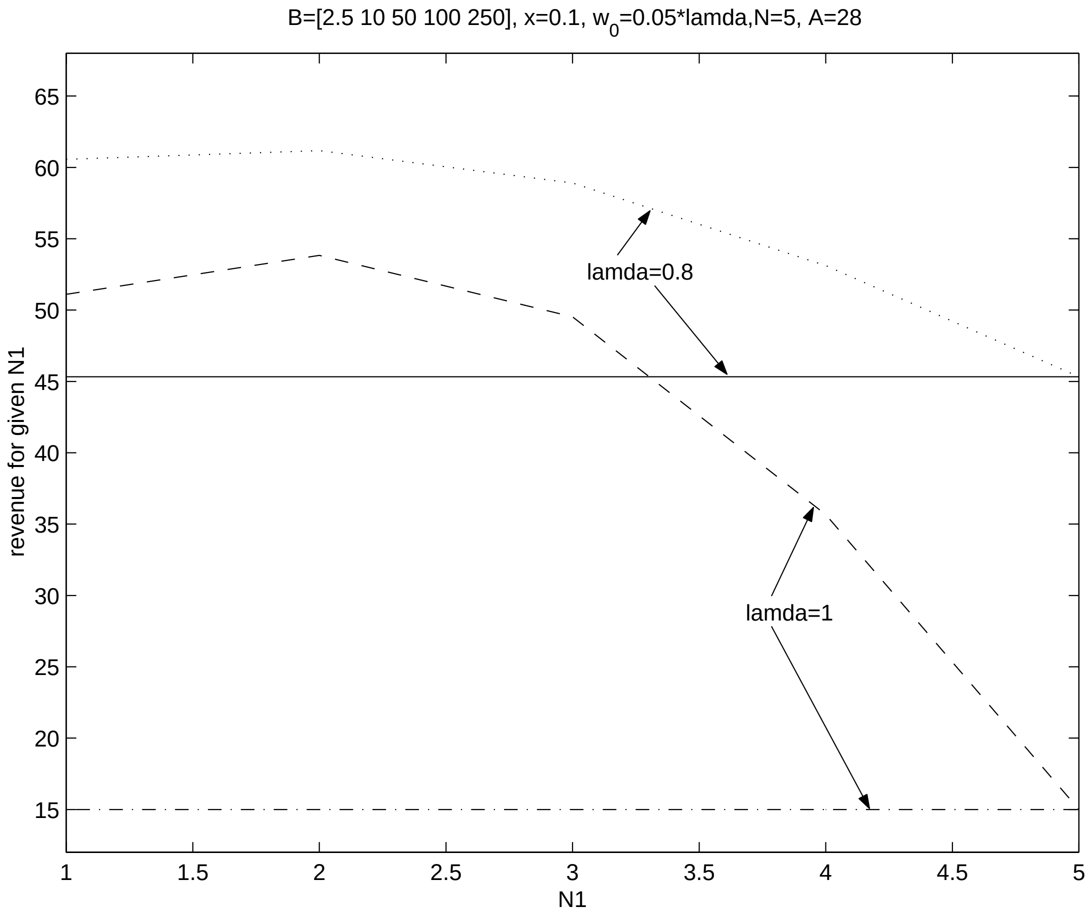

Figure 4 effect of user's traffic utilization

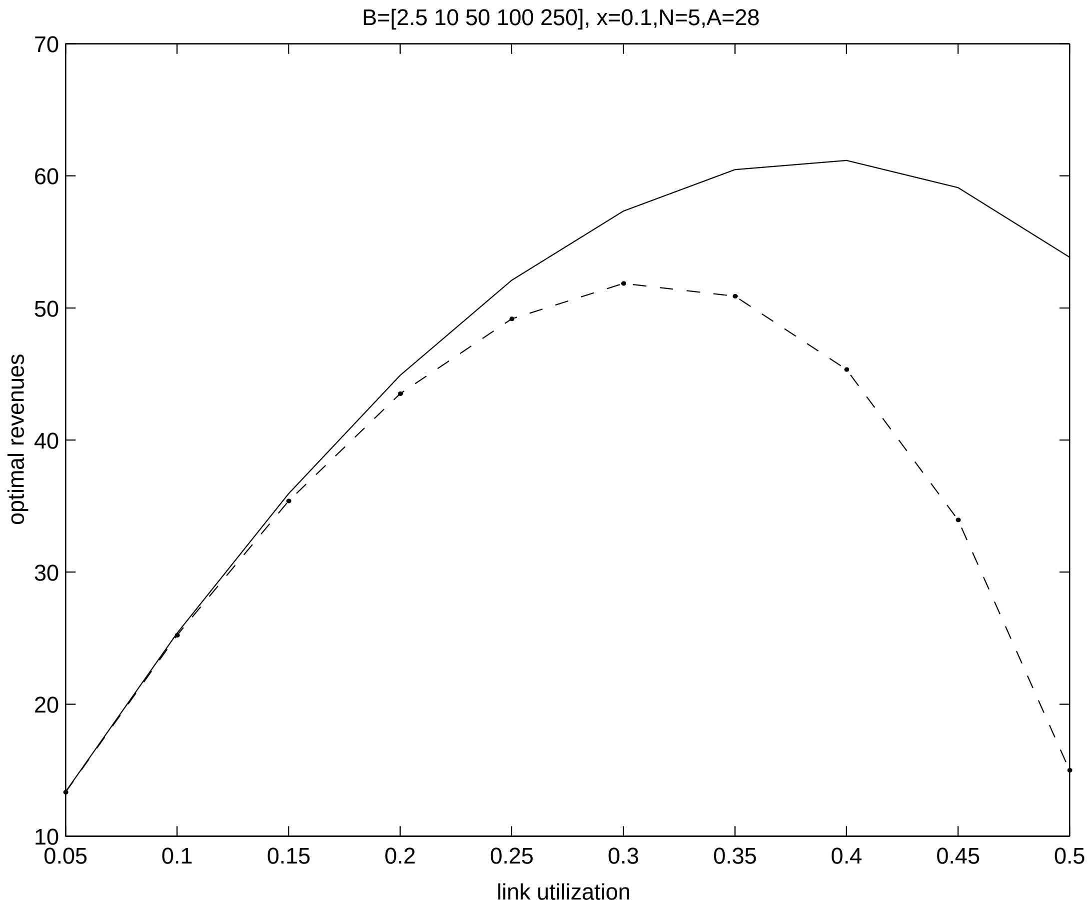

Figure 5 Optimal revenue vs. user's traffic utilization

Reference:

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5. IETF RSVP Working Group home page: http://www.isi.edu/div7/rsvp/ietf.html.

6. IETF DIFFSERV Working Group home page:
   http://www.ietf.org/html.charters/diffserv-charter.html

Appendix

The Derivation of the optimal unit prices under differential pricing scheme

We use the following notation:

B_{1max} is the maximum value of B_i among the users choosing high priority class,

B_{1min} is the minimum value of B_i among the users requesting high priority class,

B_{2max} is the maximum value of B_j among the users choosing low priority class.

$p_{1max} = A - B_{1max} W_1$,

$p_{2max} = A – B_{2max} W_2$,

$(p_1 - p_2)_{max} = B_{1min} (W_{2,+i} - W_1)$,

$(p_1 - p_2)_{min} = B_{2max} (W_{2,} - W_{1, +j})$,

The optimal uniform pricing policy revenue is :

$$N\lambda(A - B_{\max} W) \qquad \text{-- E.1}$$

where: $W = \dfrac{W_0}{1 - N\lambda x}$

and $W_0 = \frac{1}{2}(N_1\lambda\overline{x^2} + N_2\lambda\overline{x^2}) = \frac{1}{2}(N\lambda\overline{x^2})$

For the differential pricing scheme, using the above notation the optimal problem could be rewritten as follows:

Maximize: $$\sum_{i}^{N_1} p_1\lambda + \sum_{j}^{N_2} p_2\lambda = N_1 p_1\lambda + N_2 p_2\lambda$$

Subject to: $$p_1 \le A - B_{1\max} W_1 \qquad \supseteq$$

$$p_2 \le A - B_{2\max} W_2 \qquad \not\subset$$

$$(p_1 - p_2) \le B_{1\min}(W_{2,+j} - W_1) \qquad \subset$$

$$(p_1 - p_2) \ge B_{2\max}(W_2 - W_{1,+i}) \qquad \subseteq$$

where:

$$W_1 = \frac{W_0}{(1 - N_1\lambda x)}$$

$$W_2 = \frac{W_0}{(1 - N_1\lambda x)(1 - N\lambda x)}$$

$$W_{1,+i} = \frac{W_0}{[1 - (N_1 + 1)\lambda x]}$$

$$W_{2,+j} = \frac{W_0}{[1 - (N_1 - 1)\lambda x](1 - N\lambda x)}$$

For a particular N_1, the objective function is increasing in p_1 and p_2, so the optimal values of p_1 and p_2 are the maximum possible values that satisfy the constraints ⊇, ⊄, ⊂ and ⊆. Direct upper bounds for p_1 and p_2 are given by ⊇ and ⊄, while ⊂ and ⊆ provide bounds on the difference (p_1 - p_2).

For use in the following derivations, we provide the following identities:

$$p_{1\max} - p_{2\max} = A - B_{1\max}\frac{W_0}{1-N_1x\lambda} - A + B_{2\max}\frac{W_0}{(1-N_1x\lambda)(1-Nx\lambda)}$$

$$= \frac{B_{2\max}W_0 - B_{i\max}W_0(1-Nx\lambda)}{(1-N_1x\lambda)(1-Nx\lambda)}$$

$$(p_1 - p_2)_{\max} = \{\frac{Nx\lambda - \lambda x - (N_1-1)N\lambda^2x^2}{(1-N_1x\lambda)(1-Nx\lambda)[1-(N_1-1)x\lambda]}\}B_{1\min}W_0$$

$$(p_1 - p_2)_{\min} = \{\frac{Nx\lambda - \lambda x - N_1N\lambda^2x^2}{(1-N_1x\lambda)(1-Nx\lambda)[1-(N_1+1)x\lambda]}\}B_{2\max}W_0$$

We begin by showing that $B_{1min} \geq B_{2max}$. To begin, note that for conditions ⊂ and ⊆ to hold at the same time, the following inequality must hold:

$(p_1 - p_2)_{\max} \geq (p_1 - p_2)_{\min}$

i.e., $B_{1\min}(W_{2,+j} - W_1) \geq B_{2\max}(W_2 - W_{1,+i})$

This could be rewritten as:

$$\{\frac{Nx\lambda - \lambda x - (N_1-1)N\lambda^2x^2}{(1-N_1x\lambda)(1-Nx\lambda)[1-(N_1-1)x\lambda]}\}B_{1\min}W_0 \geq$$

$$\{\frac{Nx\lambda - \lambda x - N_1N\lambda^2x^2}{(1-N_1x\lambda)(1-Nx\lambda)[1-(N_1+1)x\lambda]}\}B_{2\max}W_0$$

Canceling common factors on each side of this inequality, we have:

$$\{\frac{N-1-(N_1-1)N\lambda\ x}{[1-(N_1-1)x\lambda]}\}B_{1\min} \geq \frac{N-1-N_1 N\lambda x}{[1-(N_1+1)x\lambda]}B_{2\max}$$

$$\Rightarrow \frac{N[1-(N_1-1)\lambda x]-1}{[1-(N_1-1)x\lambda]}B_{1\min} \geq \frac{N[1-(N_1+1)\lambda x]-1+N\lambda x}{1-(N_1+1)x\lambda}B_{2\max}$$

$$\Rightarrow \{N-\frac{1}{[1-(N_1-1)x\lambda]}\}B_{1\min} \geq \{N-\frac{1-Nx\lambda}{1-(N_1+1)x\lambda}\}B_{2\max}$$

We now show that B_{1min} is greater than or equal to B_{2max}. This leads to $B_{1max} \geq B_{1min} \geq B_{2max}$, implying $B_{1max} \geq B_{2max}$.

Proof:

Assume B_{1min} is less than B_{2max}, then we have:

$$\{N-\frac{1}{[1-(N_1-1)x\lambda]}\} > \{N-\frac{1-Nx\lambda}{1-(N_1+1)x\lambda}\} \Rightarrow \frac{1}{[1-(N_1-1)x\lambda]} < \frac{1-Nx\lambda}{1-(N_1+1)x\lambda}$$

$$\Rightarrow 1-(N_1+1)\lambda x < 1-(N_1-1)x\lambda - Nx\lambda + N(N_1-1)\lambda^2 x^2$$

$$\Rightarrow N_1 > 1+\frac{N-2}{N\lambda x}$$

Since $1-N\lambda x > 0$, so $N\lambda x < 1$, then:

$$1+\frac{N-2}{N\lambda x} > 1+N-2 = N-1$$

Thus, from the assumption we get $N_1 > N-1$, which is not true.

So B_{1min} is greater than or equal to B_{2max}.

We now consider the three possible cases for the value of $p_{1max} - p_{2max}$.

Case 1:

If $(p_1 - p_2)_{min} \le p_{1max} - p_{2max} \le (p_1 - p_2)_{max}$, both unit prices could reach their maximum possible value, so the optimal prices are $p_{1optimal} = p_{1max}$ and $p_{2optimal} = p_{2max}$ (see figure A.1).

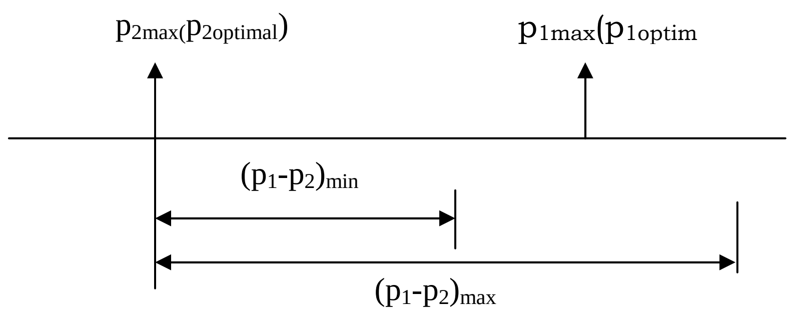

Figure A.1

Appropriately adjusting and combining the limits on B_{2max} from case 2 and 3 below, this condition can be rewritten as:

$$B_{1\max}(1-Nx\lambda)+B_{1\min}\lambda x[N-\frac{1}{1-(N_1-1)x\lambda}] \ge B_{2\max} \ge B_{1\max}\frac{[1-(N_1+1)x\lambda]}{1-N_1x\lambda}$$

The optimal revenue for a given N_1 is:

$$N_1 p_{1optimal}\lambda+(N-N_1)p_{2optimal}\lambda = N_1 p_{1\max}\lambda+(N-N_1)p_{2\max}\lambda =$$
$$N_1[A-B_{1\max}\frac{W_0}{(1-N_1x\lambda)}]\lambda+(N-N_1)[A-B_{2\max}\frac{W_0}{(1-N_1x\lambda)(1-Nx\lambda)}]\lambda$$

--E.2

$$E.2 - E.1 = \frac{W_0 \lambda (N - N_1)(B_{1\max} - B_{2\max})}{(1 - N_1 x\lambda)(1 - Nx\lambda)}$$

Since $B_{1max} > B_{2max}$, we have E.2 –E.1 >0, *i.e.*, the revenue of differential pricing scheme is greater than that of the uniform pricing scheme.

Case 2:

If $p_{1max} - p_{2max} < (p_1 - p_2)_{min}$, the optimal value of p_2 could not reach the maximum possible value, so the optimal prices are: $p_{1optimal} = p_{1max}$ and $p_{2optimal} = p_{1max} - (p_1 - p_2)_{min}$ (see figure A.2).

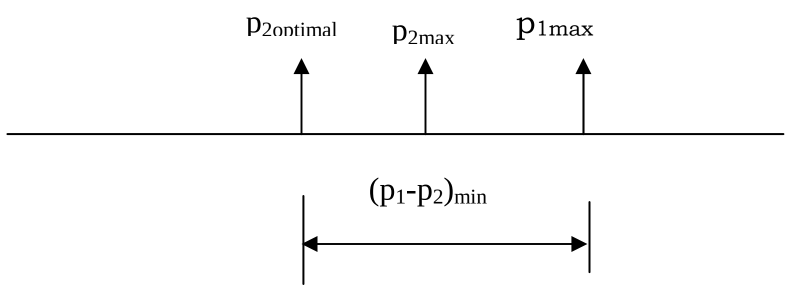

Figure A.2

The condition $p_{1max} - p_{2max} < (p_1 - p_2)_{min}$ could be rewritten as:

$$B_{2\max} < B_{1\max} [\frac{1 - (N_1 + 1)\lambda x}{1 - N_1 \lambda x]}]$$

The optimal revenue for a given N_1 is:

$$N_1 p_{1optimal}\lambda + (N - N_1)p_{2optimal}\lambda = N_1 p_{1\max}\lambda + (N - N_1)[p_{1\max} - (p_1 - p_2)_{\min}]\lambda =$$

$$N\lambda[A - B_{1\max}\frac{W_0}{(1 - N_1 x\lambda)}] - (N - N_1)\lambda B_{2\max}W_0 \frac{Nx\lambda - \lambda x - N_1 N\lambda^2 x^2}{(1 - N_1 x\lambda)(1 - Nx\lambda)[1 - (N_1 + 1)x\lambda]}$$

--E.3

E.3 –E.1=

$$\frac{\lambda(N - N_1)W_0\lambda x}{(1 - N_1 x\lambda)(1 - Nx\lambda)[1 - (N_1 + 1)x\lambda]}\{B_{1\max}N[1 - (N_1 + 1)x\lambda] - B_{2\max}[N(1 - N_1 x\lambda) - 1]\}$$

$$= \frac{\lambda(N - N_1)W_0\lambda x}{(1 - N_1 x\lambda)(1 - Nx\lambda)[1 - (N_1 + 1)x\lambda]}\left\{N\{B_{1\max}[1 - (N_1 + 1)x\lambda] - B_{2\max}(1 - N_1 x\lambda)\} + B_{2\max}\right\}$$

According to the condition, $\mathrm{B}_{1\max}[1 - (\mathrm{N}_1 + 1)\mathrm{x}\,\lambda] > \mathrm{B}_{2\max}(1 - N_1 x\lambda)$, so we have:

$$N\{B_{1\max}[1 - (N_1 + 1)x\lambda] - B_{2\max}(1 - N_1 x\lambda)\} + B_{2\max} > 0$$

Since $N > N_1$, so we have E.3-E.1 >0, *i.e.*, the revenue raised by differential pricing scheme is again greater than the uniform pricing scheme.

Case 3:

If $p_{1max} - p_{2max} > (p_1 - p_2)_{max}$, p_1 and p_2 could not take the maximum possible values at the same time. Furthermore, if p_1 is greater than $p_{2max} + (p_1 - p_2)_{max}$, then there is no possible value of p_2 that satisfies $\subset$ (see figure A.3). So $p_{1optimal} = p_{2max} + (p_1 - p_2)_{max}$ and $p_{2optimal} = p_{2max}$.

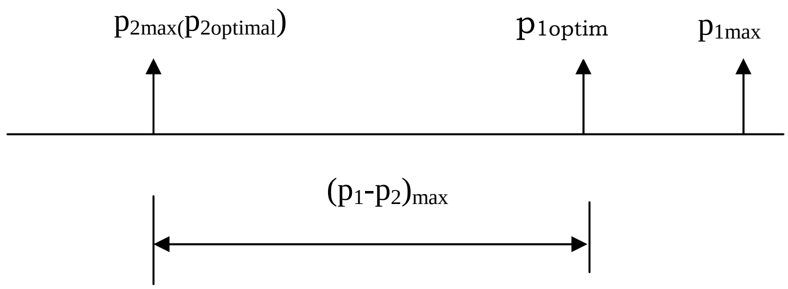

Figure A.3

The condition p_{1max} - p_{2max} > (p_1 - p_2)$_{max}$ could be rewritten as:

$$\frac{B_{2\max}W_0 - B_{1\max}W_0(1-Nx\lambda)}{(1-N_1x\lambda)(1-Nx\lambda)} >$$
$$\{\frac{Nx\lambda - \lambda x - (N_1-1)N\lambda^2x^2}{(1-N_1x\lambda)(1-Nx\lambda)[1-(N_1-1)x\lambda]}\}B_{1\min}W_0$$
$$\Rightarrow B_{2\max} > B_{1\max}(1-Nx\lambda) + B_{1\min}\lambda x[N - \frac{1}{[1-(N_1-1)x\lambda]}]$$

❺

The optimal revenue under the differential scheme for a given N_1 is:

$$N_1 p_{1optimal}\lambda + (N-N_1)p_{2optimal}\lambda = Np_{2\max}\lambda + N_1(p_1 - p_2)_{\max}\lambda$$
$$= N\lambda[A - B_{2\max}\frac{W_0}{(1-Nx\lambda)(1-N_1x\lambda)}] + N_1\lambda B_{1\min}W_0\{\frac{Nx\lambda - x\lambda - (N_1-1)Nx^2\lambda^2}{(1-Nx\lambda)(1-N_1x\lambda)[1-(N_1-1)x\lambda]}\}$$

---E.4

E.4-E.1 =

$$\{NB_{1\max}(1-N_1x\lambda)[1-(N_1-1)x\lambda]-NB_{2\max}[1-(N_1-1)x\lambda]$$
$$+N_1B_{1\min}x\lambda\{N[1-(N_1-1)x\lambda]-1\}\}\frac{W_0\lambda}{(1-N_1x\lambda)(1-Nx\lambda)[1-(N_1-1)x\lambda]}$$
$$=\frac{W_0\lambda[1-(N_1-1)x\lambda]}{(1-Nx\lambda)(1-N_1x\lambda)[1-(N_1-1)x\lambda)}\{NB_{1\max}(1-N_1x\lambda)-NB_{2\max}$$
$$+N_1B_{1\min}x\lambda[N-\frac{1}{1-(N_1-1)x\lambda}]\}$$

If $\mathrm{B}_{2\max} < \mathrm{B}_{1\max}\ (1\text{-}\ \mathrm{N}_1\ \lambda\ x) + \mathrm{B}_{1\min}\ \lambda x \frac{N_1}{N}\{\mathrm{N}-\frac{1}{1-(N_1-1)\lambda x}\}$ ❻ ,

then E.4 –E.1 >0.

The right hand side of ❻ can be rewritten as

$\mathrm{B}_{1\max}\ \text{-}\ (\mathrm{B}_{1\max}\ \text{-}\ \mathrm{B}_{1\min})\mathrm{N}_1\ \lambda\ x - \mathrm{B}_{1\min}\ \lambda x \frac{N_1}{N[1-(N_1-1)\lambda x]}\}$, which is less than B_{1max}, so it is possible for the uniform pricing scheme to produce more revenue than the differential pricing scheme.

It is possible for both ❺ and ❻ to be satisfied since (see proof below)

$$\mathrm{B}_{1\max}\ (1\text{-}\ \mathrm{N}_1\lambda\ x) + \mathrm{B}_{1\min}\ \lambda x\{\mathrm{N}_1-\frac{N_1}{N[1-(N_1-1)\lambda x]}\} >$$
$$B_{1\max}(1-Nx\lambda)+B_{1\min}\lambda x[N-\frac{1}{1-(N_1-1)x\lambda}]$$ ❼

so the result is: if $\mathrm{B}_{2\max} < \mathrm{B}_{1\max}\ (1\text{-}\ \mathrm{N}_1\ \lambda\ x) + \mathrm{B}_{1\min}\ \lambda x \frac{N_1}{N}\{\mathrm{N}-\frac{1}{1-(N_1-1)\lambda x}\}$, the revenue of differential pricing scheme is greater than uniform pricing scheme; otherwise the uniform pricing scheme is better.

Proof: Subtracting the right hand side of ❼ from the left hand side, we have:

$$\mathrm{B}_{1\max}(1\text{-}\mathrm{N}_1\lambda x)-\mathrm{B}_{1\max}(1\text{-}\mathrm{N}\lambda x)+\mathrm{B}_{1\min}x\lambda\{N_1-\frac{N_1}{N[1-(N_1-1)\lambda x\}}\}-\mathrm{B}_{1\min}\lambda x[\mathrm{N}-\frac{1}{1-(N_1-1)\lambda x}]=$$

$$B_{1\max}(N-N_1)\lambda x+B_{1\min}\lambda x[N_1-N+\frac{1}{1-(N_1-1)\lambda x}(1-\frac{N_1}{N})]=$$

$$B_{1\max}(N-N_1)\lambda x+B_{1\min}\lambda x(N-N_1)\{\frac{1}{N[1-(N_1-1)\lambda x]}-1\}=(N-N_1)\lambda x\{B_{1\max}-B_{1\min}+B_{1\min}\frac{1}{N[1-(N_1-1)\lambda x]}\}$$

Since $B_{1max} \geq B_{1min}$ and $N>N_1$, the left side minus right side is greater than 0, so ❼ holds.

Figure A.4 summarizes the results of this appendix.

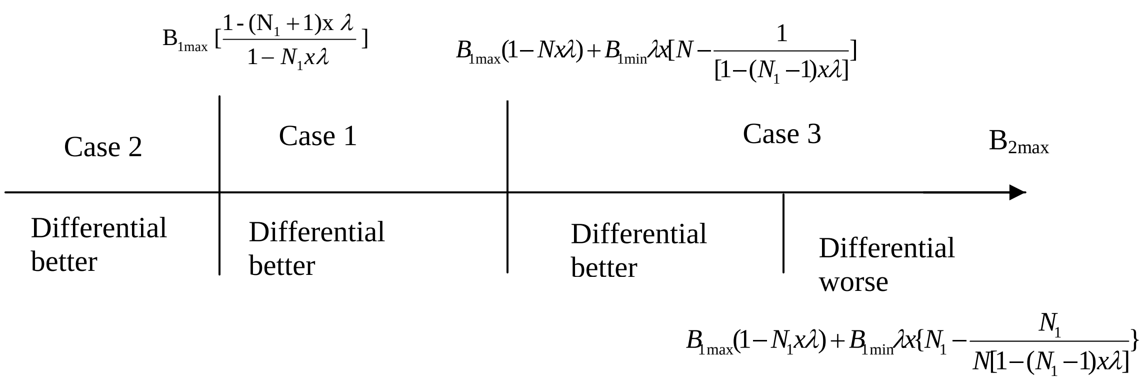

Figure A.4

From this figure, it is clear that there is a single threshold for $\mathrm{B}_{2\max}$ that determines whether differential pricing is better than uniform pricing. In conclusion, the result of comparison is:

If $\quad B_{2\max} < B_{1\max}(1-N_1\lambda x)+B_{1\min}N_1\lambda x\{1-\frac{1}{N[1-(N_1-1)\lambda x]}\}$, the revenue raised by the differential pricing scheme is greater than the uniform pricing scheme.