Synchronization Techniques for Burst-Mode Continuous Phase Modulation

Ph.D. Dissertation Defense

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Outline

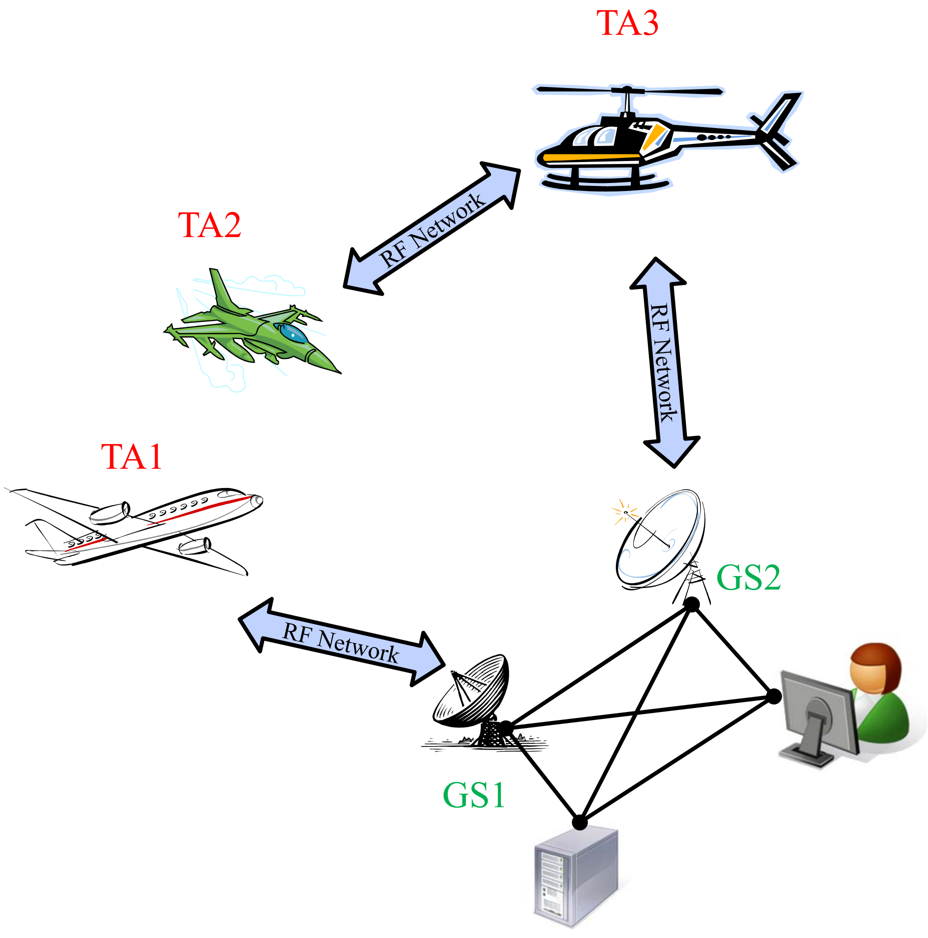
- Chapter 1: *Introduction*
- Chapter 2: *The Cramér-Rao Bound for Training Sequence Design for Burst-Mode CPM*
- Chapter 3: *Timing, Carrier and Frame Synchronization for Burst-Mode CPM*
- Chapter 4: *Applications to SOQPSK*
- Chapter 5: *Conclusions*

External Contributions

- Chapter 1: *Introduction*
- Chapter 2: *The Cramér-Rao Bound for Training Sequence Design for Burst-Mode CPM*
    - E. Hosseini and E. Perrins, "The Cramèr-Rao Bound for Training Sequence Design for Burst-Mode CPM," *IEEE Transactions on Communications*, vol. 61, no. 6, pp. 2396-2407, June 2013.
    - E. Hosseini and E. Perrins, "Training Sequence Design for Data-Aided Synchronization of Burst-Mode CPM," in Proceedings of the *IEEE Global Communications Conference (GLOBECOM'12)*, 2012.
- Chapter 3: *Timing, Carrier, and Frame Synchronization for Burst-Mode CPM*
    - E. Hosseini and E. Perrins, "Maximum Likelihood Synchronization of Burst-Mode CPM," to appear in Proceedings of the *IEEE Global Communications Conference (GLOBECOM'13),* December 9-13, 2013
    - E. Hosseini and E. Perrins, "Timing, Carrier, and Frame Synchronization of Burst-Mode CPM", *IEEE Transactions on Communications*, Under Review.
- Chapter 4: *Application to SOQPSK*
    - E. Hosseini and E. Perrins, "The Cramér-Rao Bound for Data-Aided Synchronization of SOQPSK," in Proceedings of the *IEEE Military Communications Conference (MILCOM'12)*, 2012.
    - E. Hosseini and E. Perrins, "On Burst-Mode Synchronization of SOQPSK," to appear in Proceedings of the *IEEE Military Communications Conference (MILCOM'13*), 2013.
- Chapter 5: *Conclusions*

Motivation: Aeronautical Telemetry

- integrated Network Enhanced Telemetry (iNET) project
    - Packet-based telemetry in contrast to streaming telemetry.
    - Benefits: spectrum conservation, multi-hop, enhanced mobility,..
    - A major challenge in the RF link is detection and synchronization of packets, i.e. *bursts*
    - Physical layer waveform: SOQPSK, a subset of CPM
- In this work:
    1. Synchronization for general CPM in burst-mode TX
    2. Apply the results to SOQPSK, i.e. iNET standard



GS: Ground Station
TA: Test Article

What is CPM?

- Continuous phase modulation:

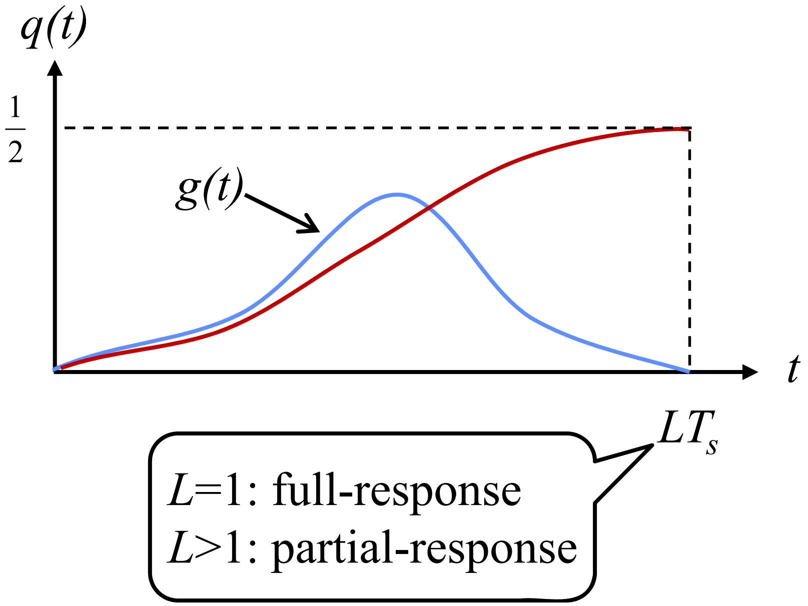
$$s(t;\boldsymbol{\alpha}) = \sqrt{\frac{E_s}{T_s}} \exp\left\{ 2\pi h \sum_i \alpha_i q(t - iT_s) \right\}$$

α_i : M-ary data symbols

h: modulation index

$q(t)$: phase response

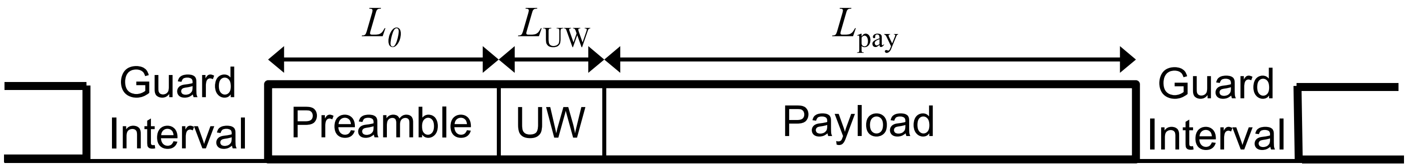
- Frequency pulse:
  - $g(t) = \dfrac{dq(t)}{dt}$
  - Duration of LT_s
- Properties:
  - Constant envelope
  - Bandwidth efficient
  - **Drawback**: complex at Rx



L=1: full-response
L>1: partial-response

Signal Transmission Model

- Burst-Mode Transmission:



L_0 L_{UW} L_{pay}

Guard Interval | Preamble | UW | Payload | Guard Interval

- Burst Structure:
  - Preamble (training sequence): sequence of L_0 known symbols
  - Unique Word (UW): burst identification
  - Payload: information bits
- Received baseband signal:

$$r(t) = e^{j(2\pi f_d t + \theta)} s(t - \tau) + w(t)$$

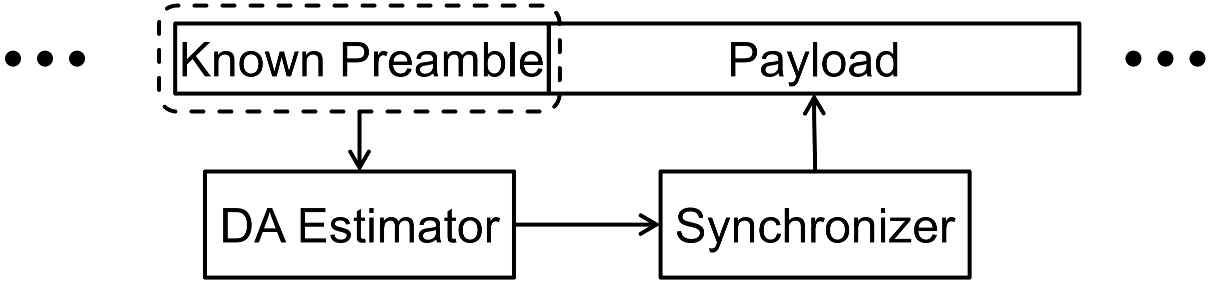
f_d : Frequency offset
θ : Carrier phase
τ : time delay

Synchronization Parameters (Unknown)

- Noise model: AWGN

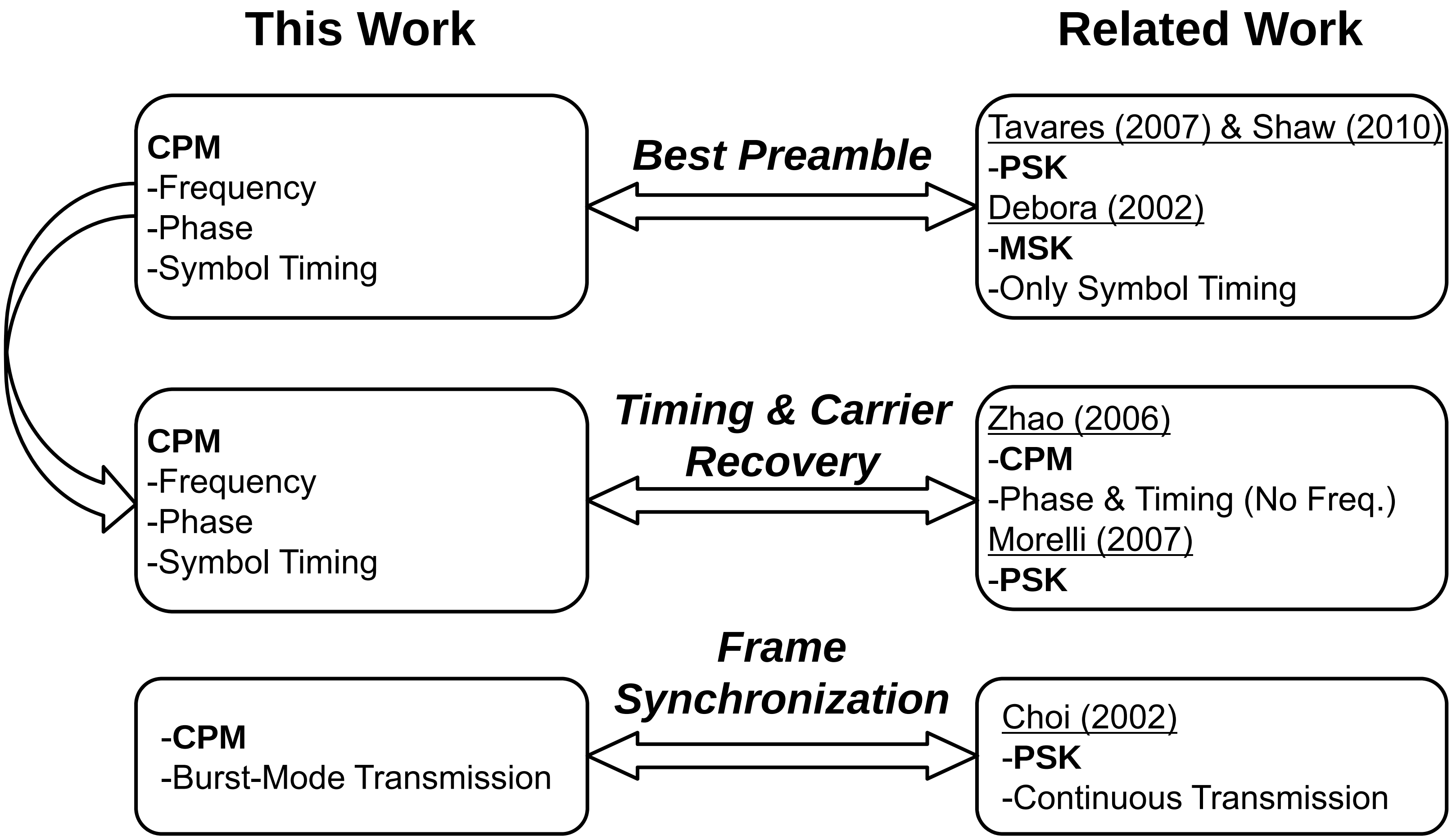
Synchronization Problem

- Challenge: Acquisition time must be kept minimum in burst-mode communications to save bandwidth, thus:
  1. Feedforward approach for shorter acquisition time
  2. Data-aided (DA) algorithms to assist synchronization



- Problems:
  1. Preamble design: What is the best preamble?
  2. Timing & carrier recovery: Estimation of $[f_d, \theta, \tau]$
  3. Frame synchronization: Detection of preamble boundaries

Comparison with Related Works



This Work

Related Work

CPM
-Frequency
-Phase
-Symbol Timing

Best Preamble

Tavares (2007) & Shaw (2010)
-PSK
Debora (2002)
-MSK
-Only Symbol Timing

CPM
-Frequency
-Phase
-Symbol Timing

Timing & Carrier Recovery

Zhao (2006)
-CPM
-Phase & Timing (No Freq.)
Morelli (2007)
-PSK

-CPM
-Burst-Mode Transmission

Frame Synchronization

Choi (2002)
-PSK
-Continuous Transmission

The Cramér-Rao Bound for Training Sequence Design for Burst-Mode CPM

Optimum Training Sequence Design: Cramér-Rao Bound Method

- Two approaches can be taken in order to derive the optimum (best) training sequence for estimation of $\mathbf{u} = \left[f_d, \theta, \tau\right]^{\mathrm{T}}$
  1. Minimize the estimation error variance:

$$\boldsymbol{\alpha}_{opt} = \arg\min_{\boldsymbol{\alpha}} E\left[\left(\hat{u}_i - u_i\right)^2 \mid \boldsymbol{\alpha}\right]$$

  2. Minimize the CRB, i.e. a theoretical lower bound on estimation error variance of *any unbiased estimator*:

$$\text{MSE} \longrightarrow E\left[\left(\hat{u}_i - u_i\right)^2 \mid \boldsymbol{\alpha}\right] \geq \mathrm{CRB}\left(u_i \mid \boldsymbol{\alpha}\right) \Rightarrow \boldsymbol{\alpha}_{\mathrm{opt}} = \arg\min_{\boldsymbol{\alpha}} \left\{\mathrm{CRB}\left(u_i \mid \boldsymbol{\alpha}\right)\right\}$$

- CRB is computed via

$$\underset{\underset{\textit{Conditional CRB}}{\uparrow}}{\mathrm{CRB}(u_i \mid \boldsymbol{\alpha})} = \mathbf{I}(\mathbf{u})_{i,i}^{-1} \quad \text{where} \quad \underset{\underset{\textit{Fisher Info. Mat. (FIM)}}{\uparrow}}{\mathbf{I}(\mathbf{u})_{i,j}} = -E\left[\frac{\partial^2}{\partial u_i \partial u_j} \ln\left(\underset{\underset{\textit{Likelihood Function}}{\nwarrow}}{p(\mathbf{r}; \mathbf{u}, \boldsymbol{\alpha})}\right)\right]$$

Overview of CRB Calculations

- Closed-form FIM for CPM and joint estimation

$$\mathbf{I}(\mathbf{u}) = \frac{1}{T_s}\left(\frac{E_s}{N_0}\right)\begin{bmatrix} 8\pi^2 T_0^3/3 & 2\pi T_0^2 & -8\pi^2 hA \\ 2\pi T_0^2 & 2T_0 & -2\pi hB \\ -8\pi^2 hA & -2\pi hB & 8\pi^2 h^2 C \end{bmatrix}$$

- It is shown the optimum training sequence must satisfy these two conditions at the same time
  1. $A' = B' = 0$

$$A' = \sum_{i=1}^{L_0} i\alpha_{i-1} = 0$$

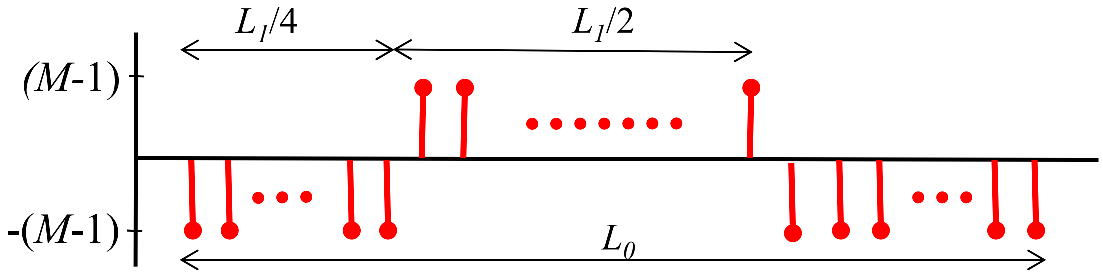
$$B' = \sum_{i=1}^{L_0} \alpha_{i-1} = 0$$

  2. Maximize C

$$C \approx R_g(0)\sum_{i=0}^{L_0-1}\alpha_i^2 + 2R_g(T_s)\sum_{i=0}^{L_0-2}\alpha_i a_{i+1} \text{ where } R_g(t) = \int_{-\infty}^{+\infty} g(u)g(u+t)du$$

Proposed Optimum Training Sequence

- Proposed sequence based on our analysis



- It satisfies $A'=B'=0$,
  1. Optimum sequence for full-response CPM.
  2. Only 2 transitions => *Near-optimum* sequence for partial-response CPM

$$(1)\, A' = \sum_{i=1}^{L_0} i\alpha_{i-1} = 0 \quad (2)\, B' = \sum_{i=1}^{L_0} \alpha_{i-1} = 0$$

$$(3)\ \text{Maximize}\ C \approx R_g(0)\sum_{i=0}^{L_0-1}\alpha_i^2 + 2R_g(T_s)\sum_{i=0}^{L_0-1}\alpha_i a_{i+1}$$

=0 for full-response CPM

- We can reduce the approximations for partial-response by using $L_1 = L_0 - \lfloor L/2 \rfloor$ rather than L_0 to reduce the *edge-effect* (EF)

Findings and Properties of the Optimum Sequence

1. The same sequence is optimum for frequency, phase, and timing, which is opposed to linear modulations.
2. It follows the same pattern for all CPMs.
3. The minimized frequency and phase CRBs are *independent* of underlying CPM parameters.
4. The minimized symbol timing CRB do depend on CPM parameters such as frequency pulse.
5. It *decouples* symbol timing estimation from phase and frequency offset estimates.

$$\mathbf{I}^*(\mathbf{u}) = \frac{1}{T_s}\left(\frac{E_s}{N_0}\right)\begin{bmatrix} 8\pi^2 T_0^3/3 & 2\pi T_0^2 & \boxed{0} \\ 2\pi T_0^2 & 2T_0 & \boxed{0} \\ \boxed{0} & \boxed{0} & 8\pi^2 h^2 C^* \end{bmatrix}$$

u_1 : frequency

u_2 : phase

u_3 : timing

More Findings

6. The proposed sequence is *asymptotically* optimum for partial-response CPMs

$$\frac{\mathrm{CRB}(\tilde{\boldsymbol{\alpha}} \mid \tau)}{\mathrm{CRB}(\boldsymbol{\alpha}^* \mid \tau)} = \frac{\tilde{C} - 8R_g(T_s)}{\tilde{C}} = 1 - \frac{8R_g(T_s)}{L_0 R_g(0) + 2(L_0 - 1)R_g(T_s)} \xrightarrow[L_0 \to \infty]{} 1$$

$\tilde{\boldsymbol{\alpha}}$: Hypothetical sequence

$\boldsymbol{\alpha}^*$: Proposed sequence

7. Computer search confirms the proposed sequence at small L_0

8. We also computed CRB for random sequences and compared with the optimum one:
   - The optimum sequence delivers significant gain for symbol timing estimation of partial-response and/or M-ary CPMs.
   - The frequency and phase estimation are less sensitive to the selection of training sequence especially for long sequences.

Timing, Carrier, and Frame Synchronization of Burst-Mode CPM

Timing and Carrier Recovery Framework

- Received Signal: $r(t) = e^{j(2\pi f_d t+\theta)} s(t-\tau) + w(t)$
- Decompose timing delay:

$$\tau = \mu T_s + \varepsilon T_s \quad \text{where } \begin{cases} \mu \geq 0 : \text{integer delay} \\ -0.5 < \varepsilon < 0.5 : \text{fractional delay} \end{cases}$$

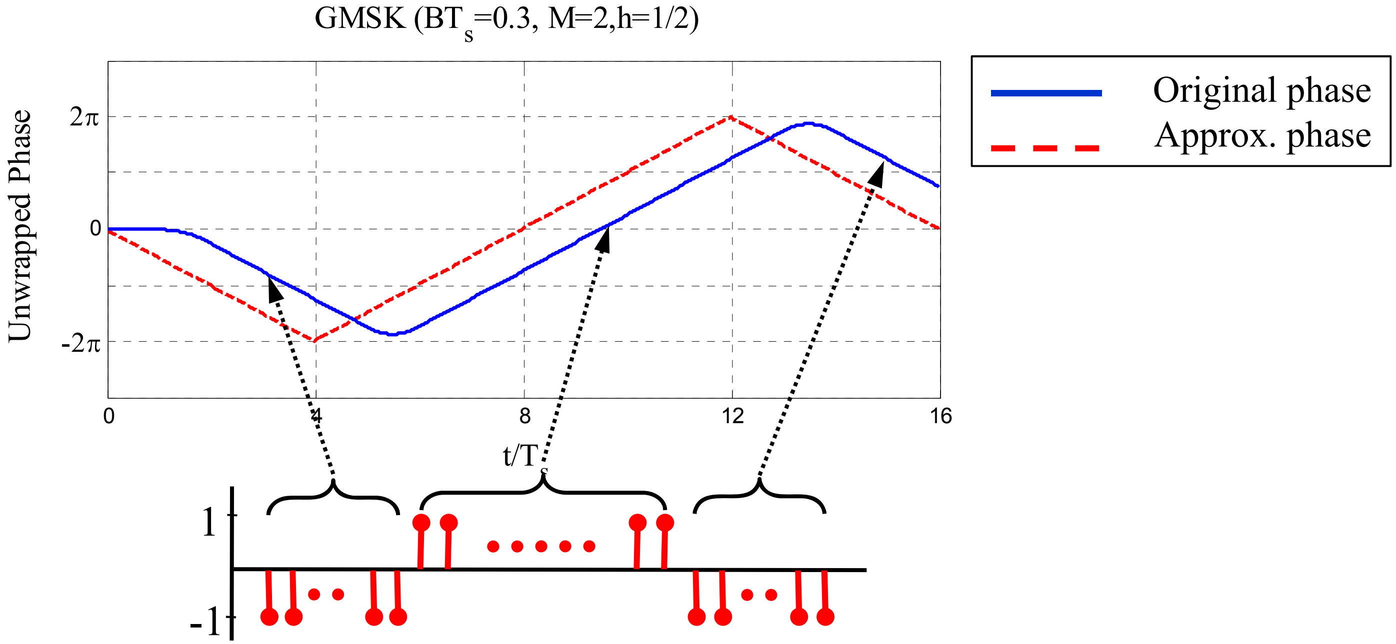
- For timing and carrier recovery, we assume integer delay is known
- Joint maximum likelihood (ML) estimator:

$$\left[\hat{f}_d, \hat{\theta}, \hat{\varepsilon}\right] = \underset{[f_d,\theta,\varepsilon]}{\arg\max} \left\{ \Lambda\left[r(t); f_d, \theta, \varepsilon\right] = \mathrm{Re}\left[\int_{\varepsilon T_s}^{T_0+\varepsilon T_s} e^{-j(2\pi f_d t+\theta)} r(t) s^*(t-\varepsilon T_s) dt \right] \right\}$$

- f_d and ε are embedded inside the above integral → It requires a 2-dimensional grid search
- **Objective**: decouple timing and frequency in the LLF

The Trick: Piecewise Linear Approximation

- Phase response of the optimum sequence



- Similar and consequent symbols cause the phase to increase/decrease uniformly
- We use linear approximation of the phase in each segment
- There is a lag for partial-response CPMs→ T_l

Approximated Log-Likelihood Function (LLF)

- Discrete-time LLF:

$$\Lambda(\mathbf{r};\nu,\theta,\varepsilon)\approx \mathrm{Re}\left[\sum_{n=0}^{NL_0-1} e^{-j(2\pi n\nu+\theta)} r[n] s_{\varepsilon}^{*}[n]\right]$$

ν: normalized freq. offset
ε: fractional delay
$s_{\varepsilon}[n]$:baseband signal shifted by ε

- Approximated baseband signal:

$$\phi(t,\boldsymbol{\alpha}^{*}) = 2\pi h \sum_{i=0}^{L_0-1} \alpha_i^{*} q(t-iT_s)$$

Original phase

Linear phase approximation

$$\phi(t,\boldsymbol{\alpha}^{*})\approx\begin{cases} -(M-1)\pi h(t-T_l)/T_s & T_l \le t < T_0/4+T_l \\ (M-1)\pi h(t-T_0/2-T_l)/T_s & T_0/4+T_l \le t < 3T_0/4+T_l \\ -(M-1)\pi h(t-T_0-T_l)/T_s & 3T_0/4+T_l \le t < T_0+T_l \end{cases}$$

Time shift + sampling + modulation

$$s_{\varepsilon}[n]\approx\begin{cases} \exp[-j(M-1)\pi h(n/N-\varepsilon)] & n \le n < NL_0/4 \\ \exp[+j(M-1)\pi h(n/N-L_0/2-\varepsilon)] & NL_0/4 \le n < 3NL_0/4 \\ \exp[-j(M-1)\pi h(n/N-L_0-\varepsilon)] & 3NL_0/4 \le n < NL_0 \end{cases}$$

will be used in the above LLF

Maximization of the LLF

- Our approximation decouples timing and frequency offsets:

$$\Lambda^{*}(\mathbf{r};\nu,\theta,\varepsilon) \approx \mathrm{Re}\left\{ e^{-j\theta}\left[e^{-j(M-1)\pi h\varepsilon}\lambda_1(\nu) + e^{j(M-1)\pi h\varepsilon}\lambda_2(\nu) \right] \right\}$$

Functions of $\mathbf{r}$

- The maximization of the LLF becomes straightforward:
  - Step 1: Frequency offset estimation:

$$\hat{\nu} = \arg\max_{\tilde{\nu}} \left\{ \left|\lambda_1(\tilde{\nu})\right| + \left|\lambda_2(\tilde{\nu})\right| \right\} \longrightarrow \text{Grid search}$$

  - Step 2: symbol timing estimation:

$$\hat{\varepsilon} = \frac{\arg\left\{ \lambda_1(\hat{\nu})\lambda_2^{*}(\hat{\nu}) \right\}}{2(M-1)\pi h}$$

  - Step 3: carrier phase estimation:

$$\hat{\theta} = \arg\{ e^{-j(M-1)\pi h\hat{\varepsilon}}\lambda_1(\hat{\nu}) + e^{j(M-1)\pi h\hat{\varepsilon}}\lambda_2(\hat{\nu}) \}$$

Implementation of Frequency Estimator

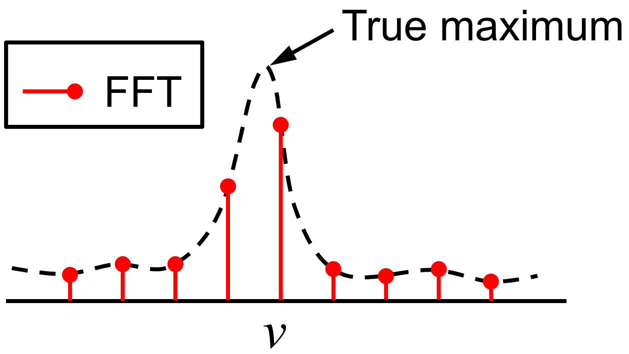
- The received $r[n]$ is *pre-processed*

$$r_1[n] = \begin{cases} r[n] & 0 \le n < NL_0/4 \\ \exp[-j(M-1)\pi hL_0]r[n] & 3NL_0/4 \le n < NL_0 \\ 0 & \text{otherwise} \end{cases} \qquad r_2[n] = \begin{cases} \exp[j(M-1)\pi hL_0/2]r[n] & NL_0/4 \le n < 3NL_0/4 \\ 0 & \text{otherwise} \end{cases}$$

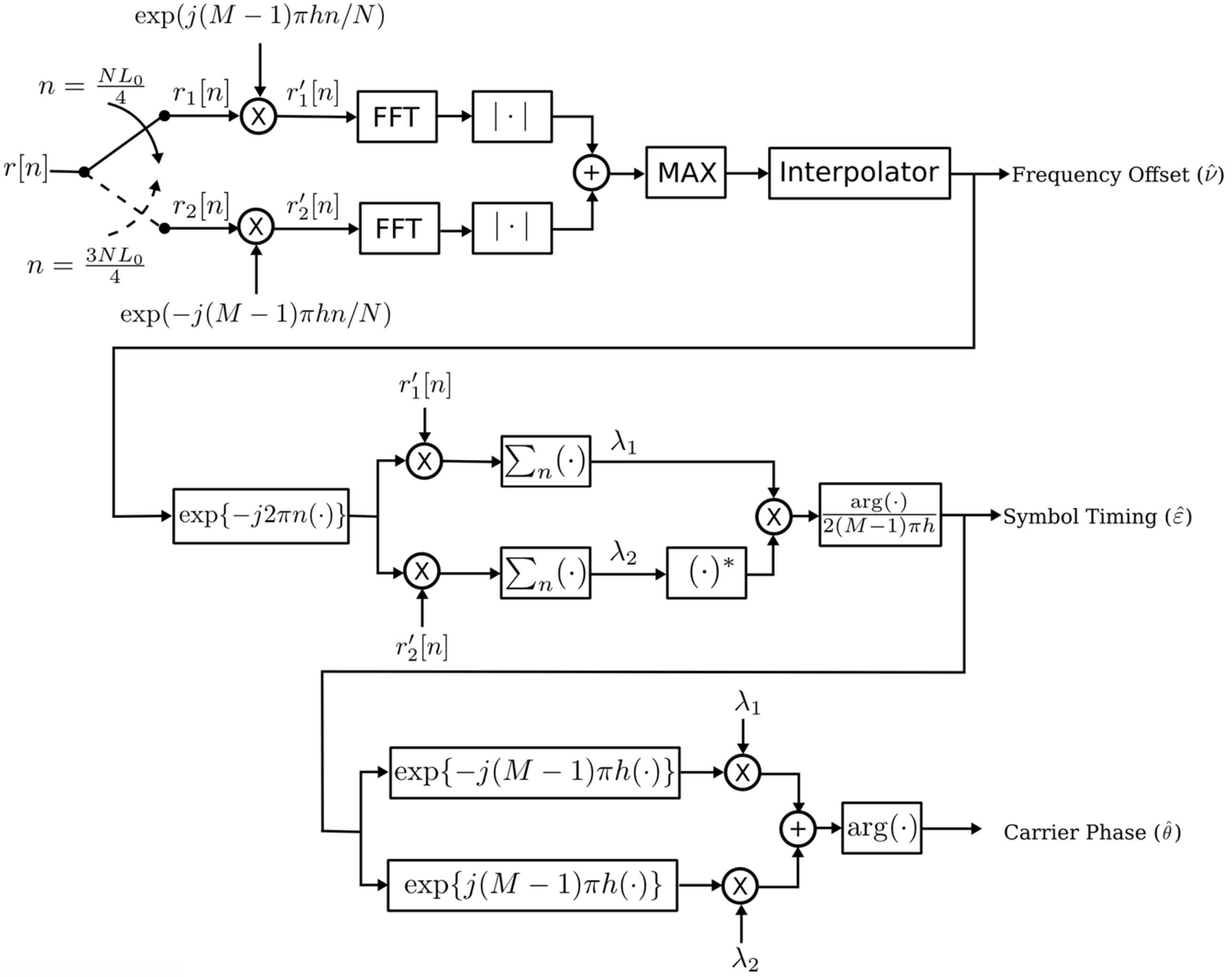
- $\lambda_1(\nu)$ and $\lambda_2(\nu)$ are computed using FFT operations:

$$\lambda_1(\nu) = \sum_{n=0}^{NL_0-1} r_1[n]e^{j(M-1)\pi hn/N}e^{-j2\pi n\nu} \qquad \lambda_2(\nu) = \sum_{n=0}^{NL_0-1} r_2[n]e^{-j(M-1)\pi hn/N}e^{-j2\pi n\nu}$$

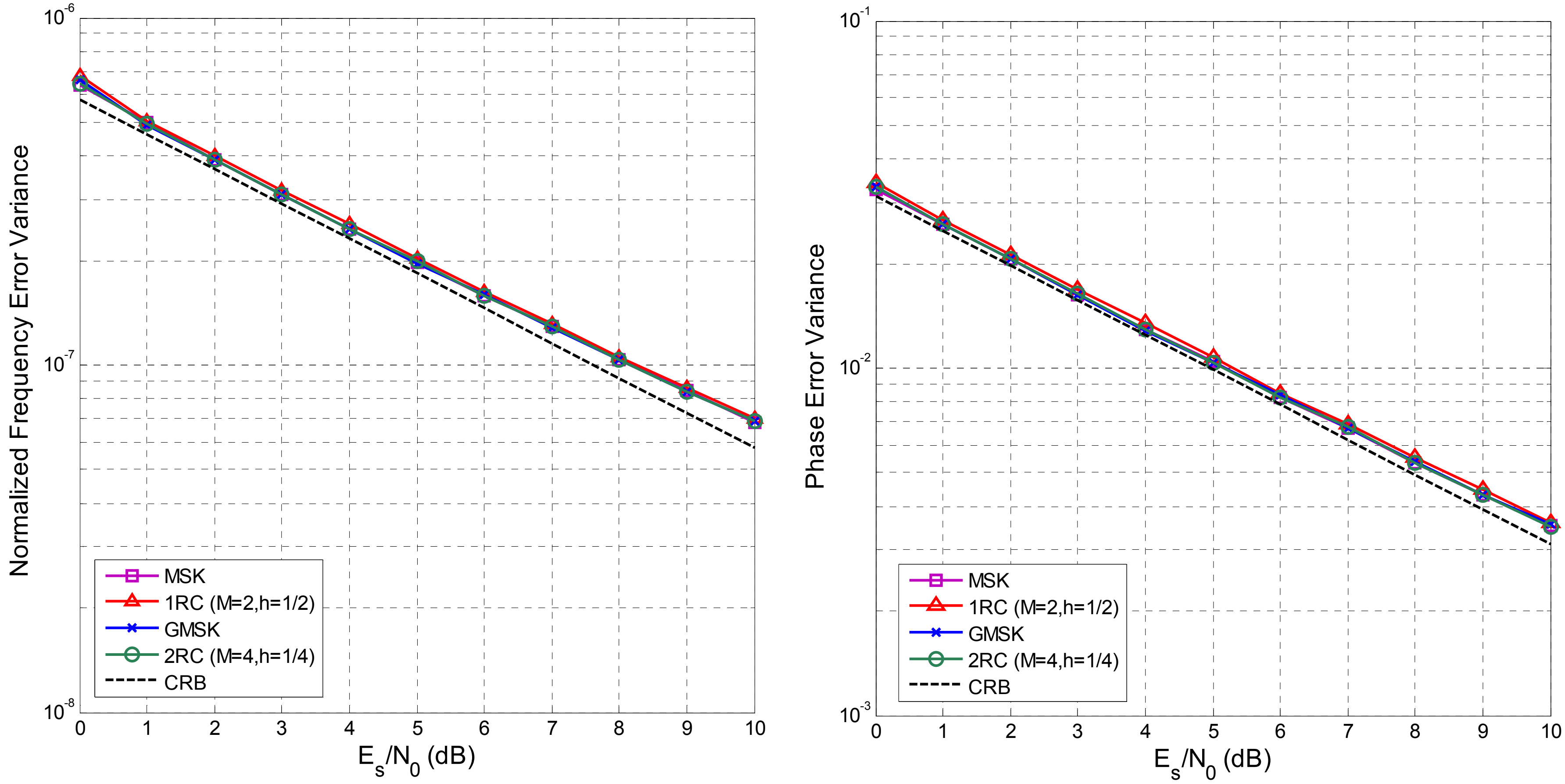
- The precision can be increased via:
  1. Zero padding prior the FFTs → K_f
  2. Interpolation of FFT results



Joint ML Estimator Block Diagram

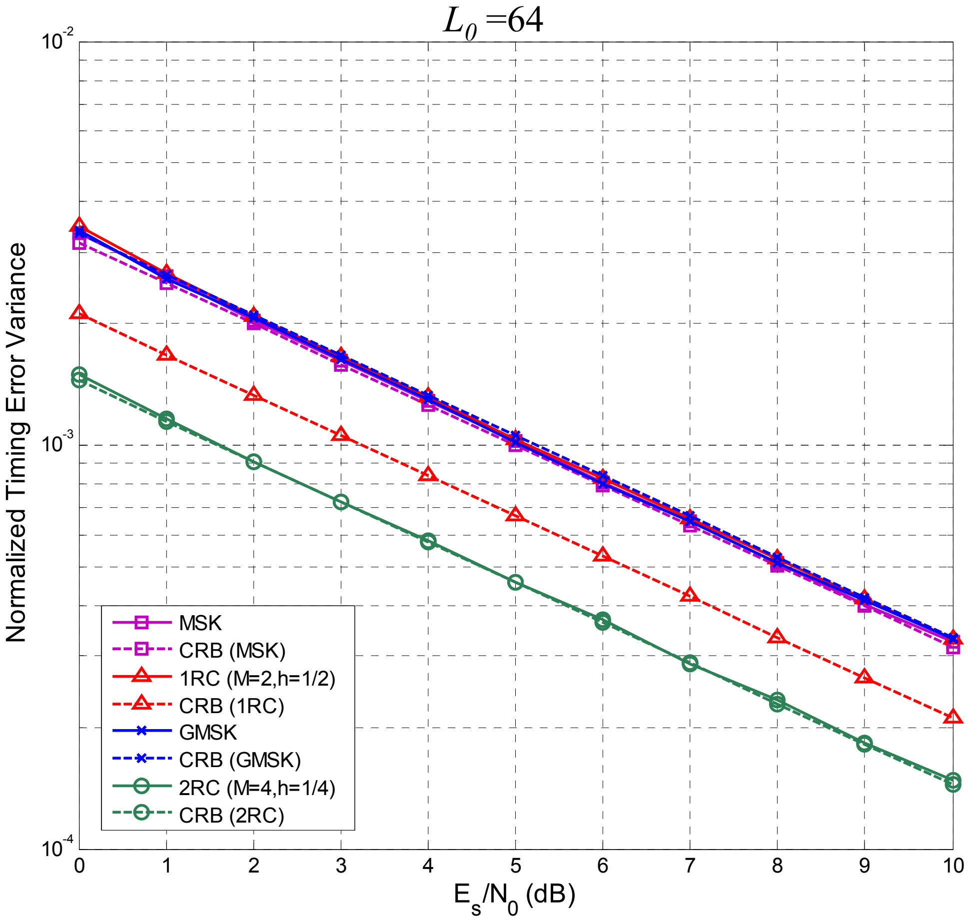


Estimator's Performance: Frequency and Phase Offsets



- Similar performance for different CPM schemes with L_0=64
- Within 0.5 dB of CRB at low to medium SNRs

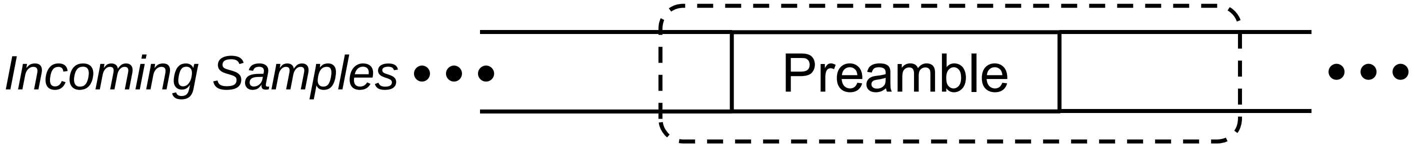
Estimator's Performance: Timing Offset



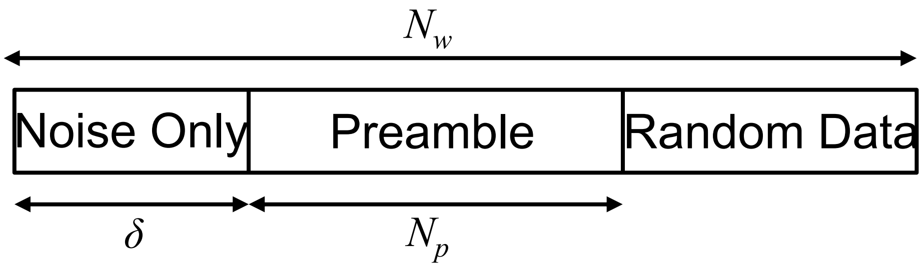
- The error variance attains the CRB for various CPMs.
- Exception: 1RC due to large deviations from our *template signal*.

Frame Synchronization: Overview

- Prior to Timing and Carrier recovery we must determine preamble boundaries, i.e. start-of-signal (SoS), in 2 steps:
  1. <u>SoS detection algorithm</u>: Detects the arrival of a new burst and *tentative* location of the SoS



  2. <u>SoS estimation algorithm</u>: Estimates the exact location of SoS within an observation window



N_w : Observation window length
N_p : Preamble length
δ : Estimation parameter

SoS Estimation Algorithm

- Apply ML estimation rules to the observation window

$$p(\mathbf{r};\delta,\nu,\theta,\boldsymbol{\alpha}_d)=\frac{1}{(\pi\sigma^2)^{N_w}}\exp\left(-\frac{1}{\sigma^2}\sum_{n=0}^{\delta-1}\left|r[n]\right|^2\right)\exp\left(-\frac{1}{\sigma^2}\sum_{n=\delta}^{N_w-1}\left|r[n]-s[n-\delta]e^{j(2\pi\nu n+\theta)}\right|^2\right)$$

- *Unwanted* parameters, [$\nu,\theta,\boldsymbol{\alpha}_d$] ,are averaged out

$$p(\mathbf{r};\delta)\approx C(\delta)\left(\sum_{n=\delta}^{N_w-1}\left|r[n]\right|^2+2\sum_{d=1}^{N_p-1}\left|\sum_{n=\delta}^{N_p+\delta-d-1}r^*[n]r[n+d]s[n-\delta]s^*[n+d-\delta]+R_{ss}(d)\sum_{n=N_p+\delta}^{N_w-d-1}r^*[n]r[n+d]\right|\right)$$

- Find δ for which the likelihood function is maximized

$$\hat{\delta}=\arg\max_{\delta} C(\delta)\left(\sum_{n=\delta}^{N_w-1}\left|r[n]\right|^2+2\sum_{d=1}^{D}\left|\sum_{n=\delta}^{N_p+\delta-d-1}r^*[n]r[n+d]s[n-\delta]s^*[n+d-\delta]+R_{ss}(d)\sum_{n=N_p+\delta}^{N_w-d-1}r^*[n]r[n+d]\right|\right)$$

$C(\delta)=(N_w-\delta)^q$ $\quad q>0:$ optimized via simulations $\qquad 1\le D<N_p$: Complexity factor

SoS Detection Algorithm

- We apply a simple likelihood ratio test (LRT)

$$L(\mathbf{r}_p) = \frac{p(\mathbf{r}_p; H_1)}{p(\mathbf{r}_p; H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$\mathbf{r}_p$: vector of N_p samples
H_0 : No preamble
H_1 : Fully-aligned preamble
γ : Test threshold

- Using the same techniques as SoS estimator, the LRT becomes

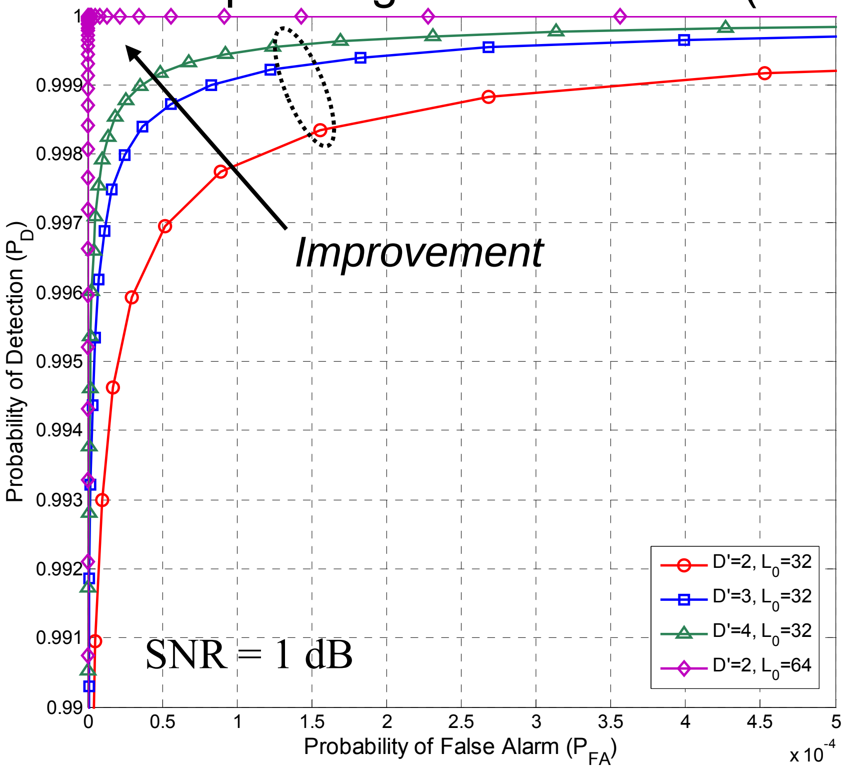
$$L_{D'}(\mathbf{r}_p) = \sum_{d=1}^{D'} \left| \sum_{n=0}^{N_p-d-1} r^*[n]r[n+d]s[n]s^*[n+d] \right| \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_{D'}$$

  - $1 \leq D' < N_p$: determines complexity
  - $\gamma_{D'}$: test threshold; selected for a given *probability of false alarm* (Neyman-Pearson criterion).
  - ROC: $P_{FA} = \Pr\{L_{D'}(r_p) > \gamma_{D'} \mid H_0\}$ vs. $P_D = 1 - \Pr\{L_{D'}(r_p) < \gamma_{D'} \mid H_1\}$
  - The LRT is performed using a sliding window.

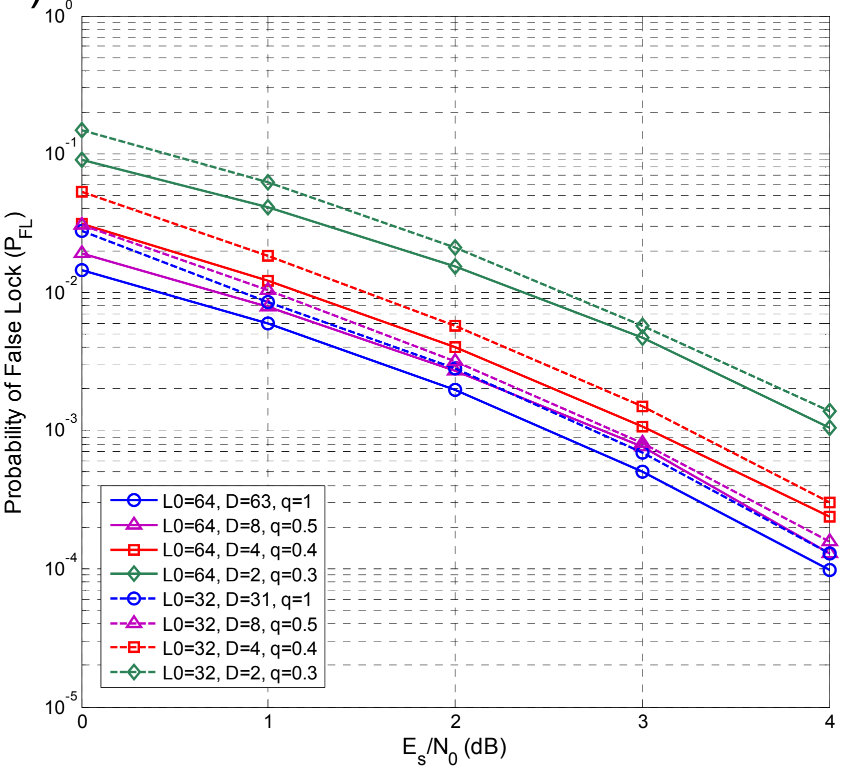
Frame Synchronization Performance

SoS Detector

Receiver Operating Characteristics (ROCs)



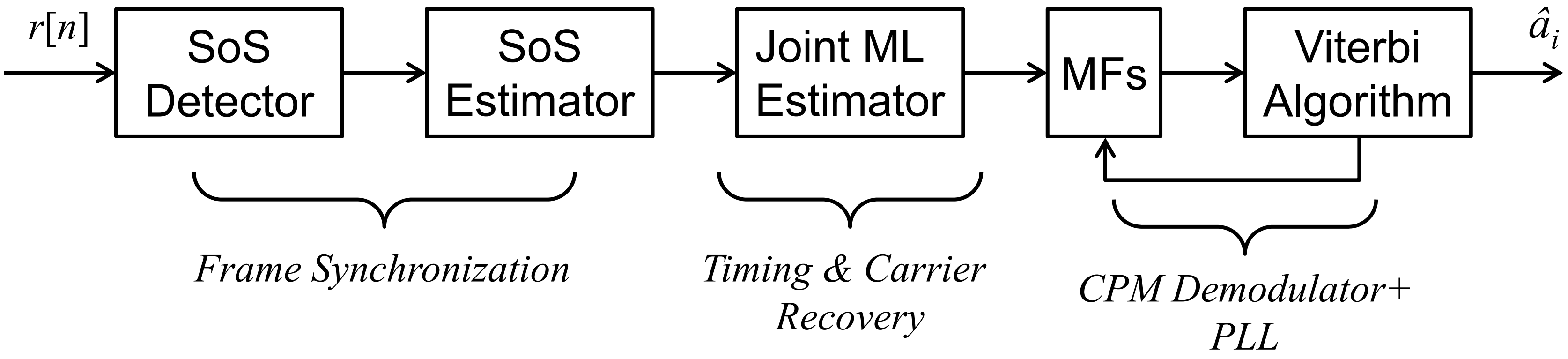
SoS Estimator



- Probabilities are computed via simulations
- Probability of false lock: $P_{FL} = \Pr\{\hat{\delta} \neq \delta\}$

Putting Everything Together: Simulating a Burst-Mode CPM Receiver

- Transmit bursts of optimum preamble (L_0) + 4096 information bits
- Apply AWGN + random frequency, phase and timing offsets
- Receiver's structure:



$r[n]$

SoS Detector

SoS Estimator

Joint ML Estimator

MFs

Viterbi Algorithm

$\hat{a}_i$

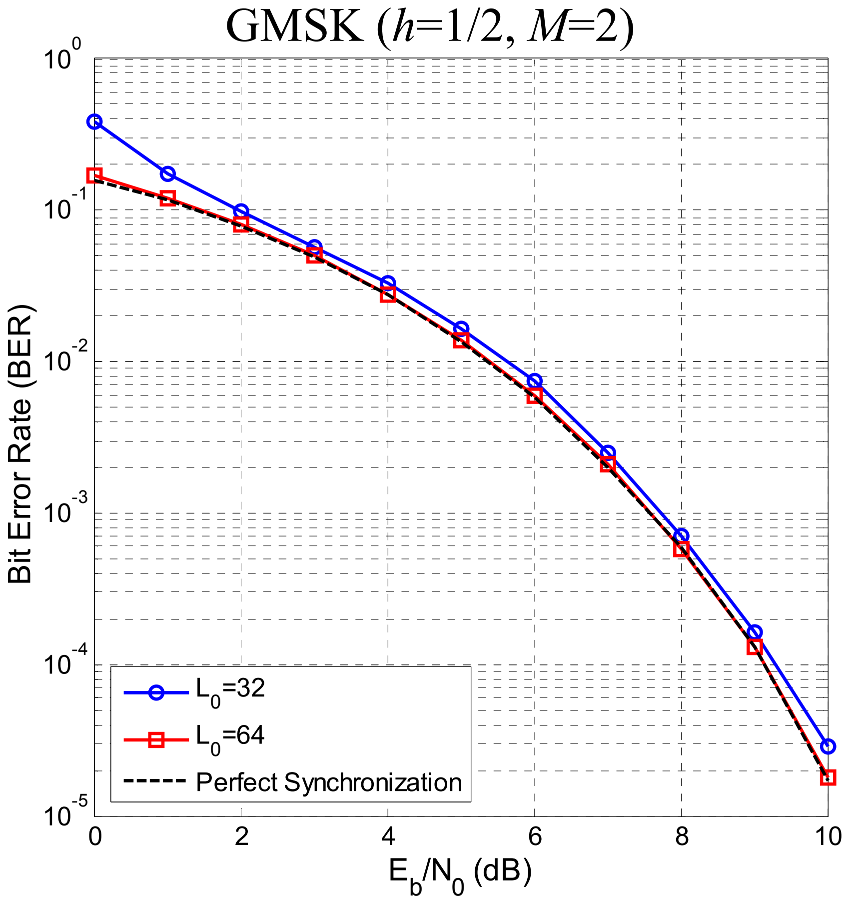
Frame Synchronization

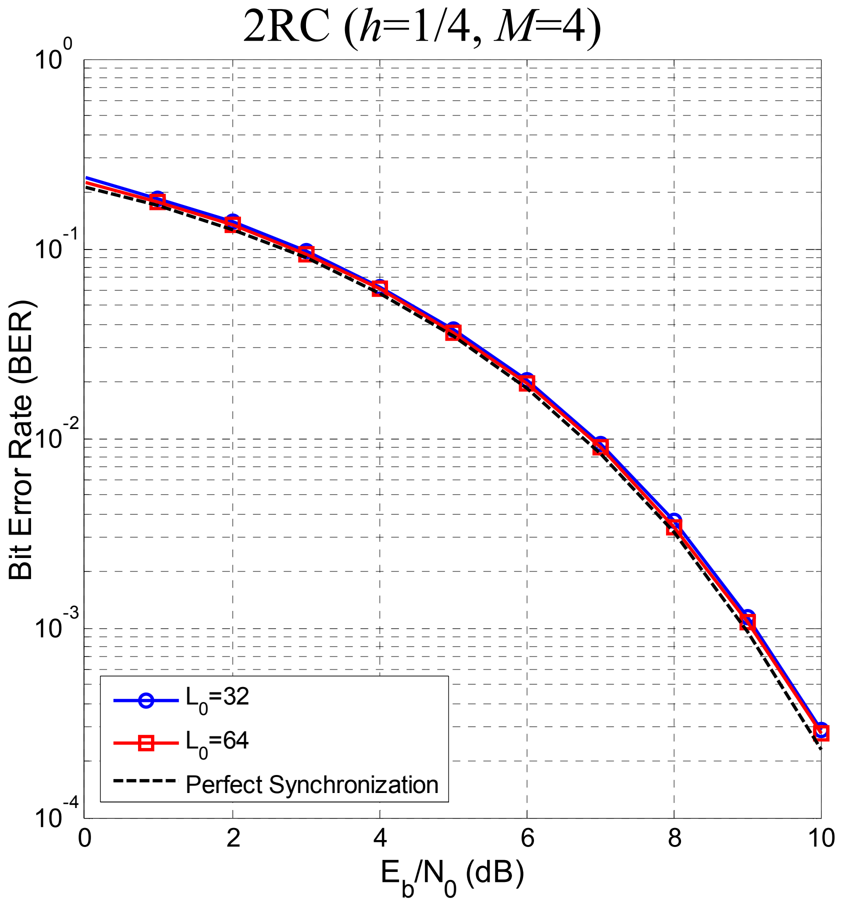
Timing & Carrier Recovery

CPM Demodulator+ PLL

- Design parameters:
  - N=2, K_f=2, Gaussian interpolator
  - D=D'=4, N_w =$2NL_0$
  - $\gamma_{D'}$ optimized for each scenario using simulations

Bit Error Rate (BER) Performance



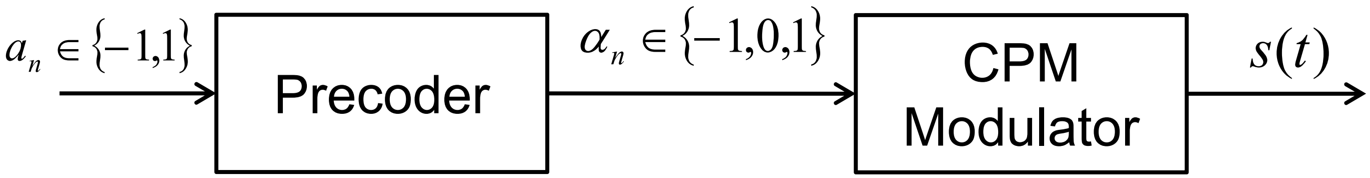


- A preamble of 64 bits performs within 0.1 dB of the ideal synchronization

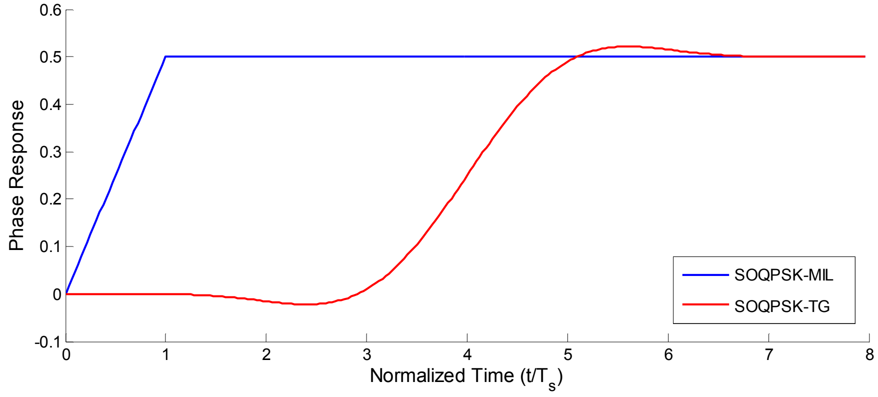
Applications to SOQPSK
(iNET)

What is SOQPSK?

- An OQSPK-like modulation, which has a CPM representation



- The precoder increases spectral efficiency of SOQPSK compared to a binary CPM by imposing:
  - $\alpha_{n-1} = 1$ cannot be followed by $\alpha_n = -1$ and vice versa
- Partial-response SOQPSK-TG is adopted by the iNET standard

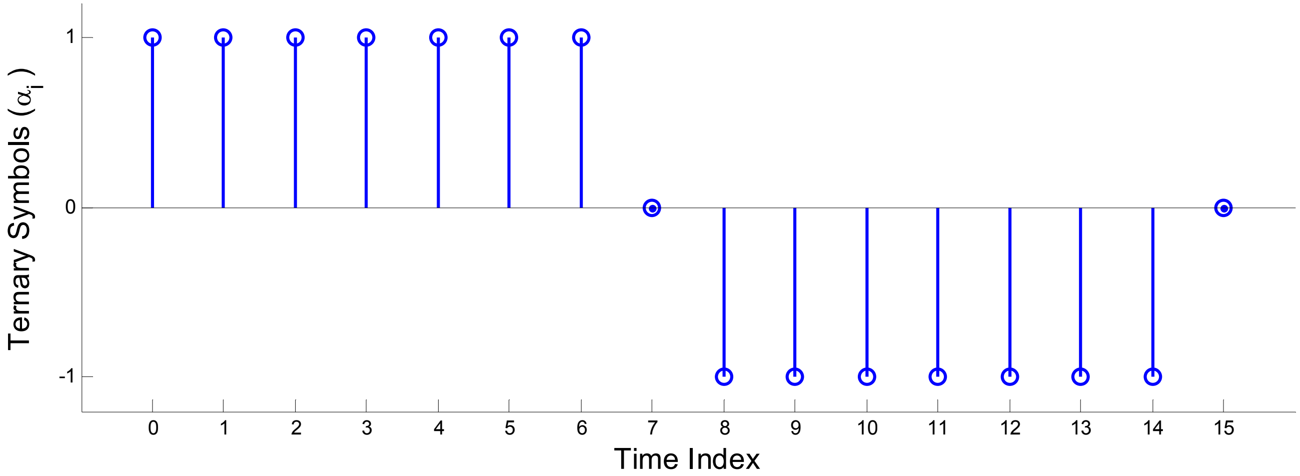


iNET Synchronization Preamble

- Length of L_0=128 bits
- Periodic and consists of repeating a sequence of 16 bits 8 times as

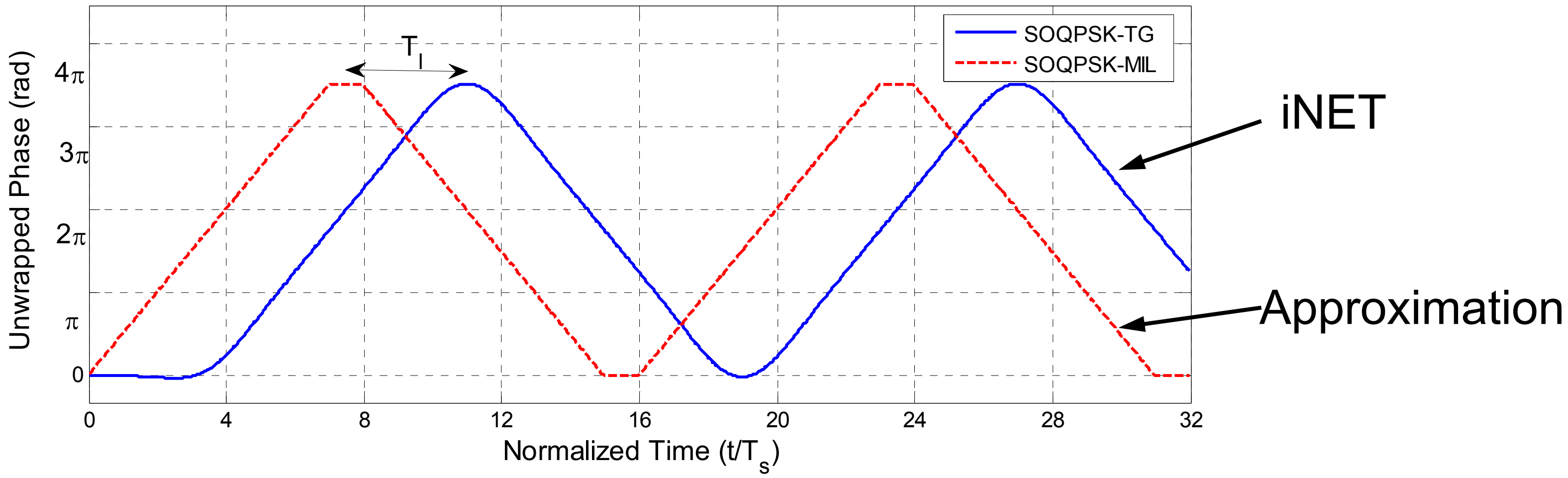
$$\left.\begin{aligned} a_{2k} &= 1,0,1,0,1,0,1,0 \\ a_{2k+1} &= 1,0,1,1,0,1,0,0 \end{aligned}\right\} \text{for } k = 0,\ldots,7$$

- One period in terms of <u>ternary symbols</u>



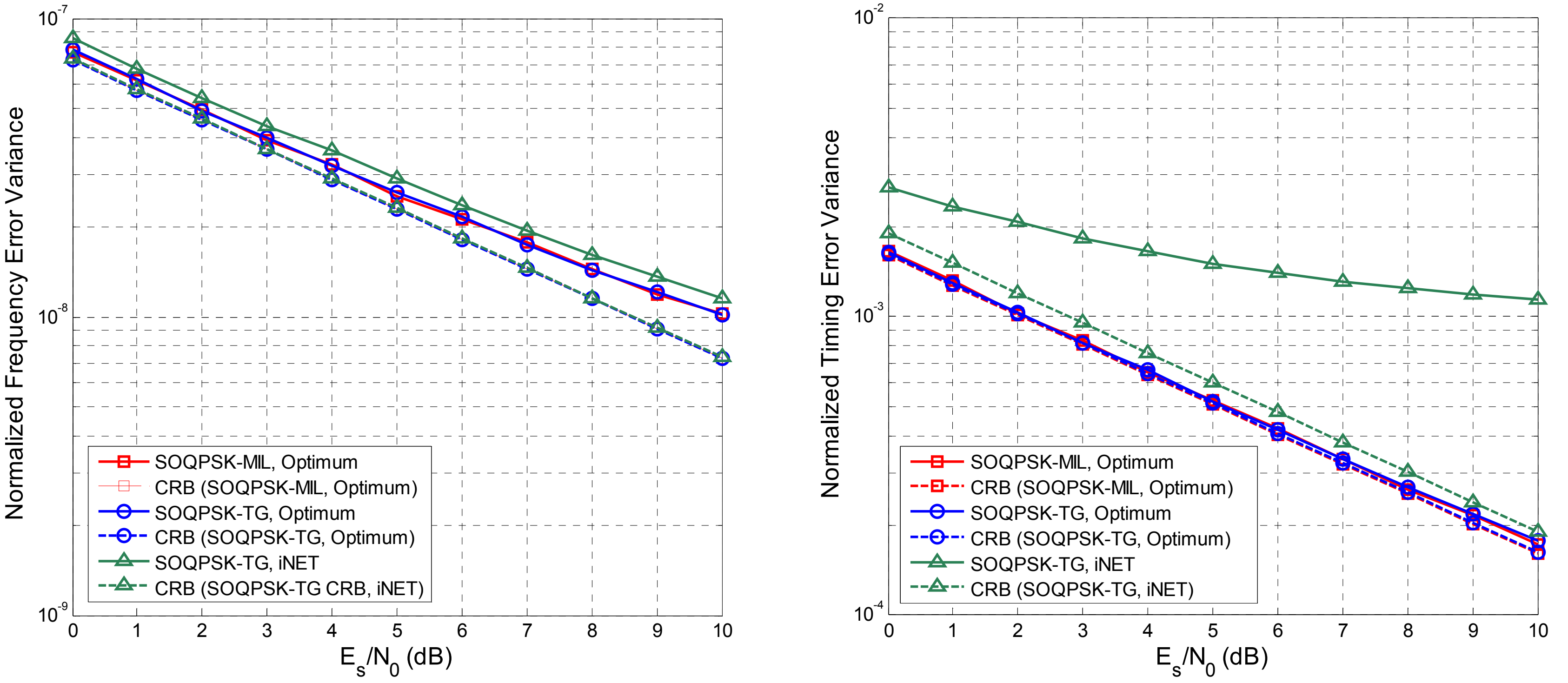
Burst-Mode Synchronization of SOQSPK-TG: iNET Preamble

- We apply our CPM techniques to SOQPSK by considering ternary symbols
- iNET preamble phase response



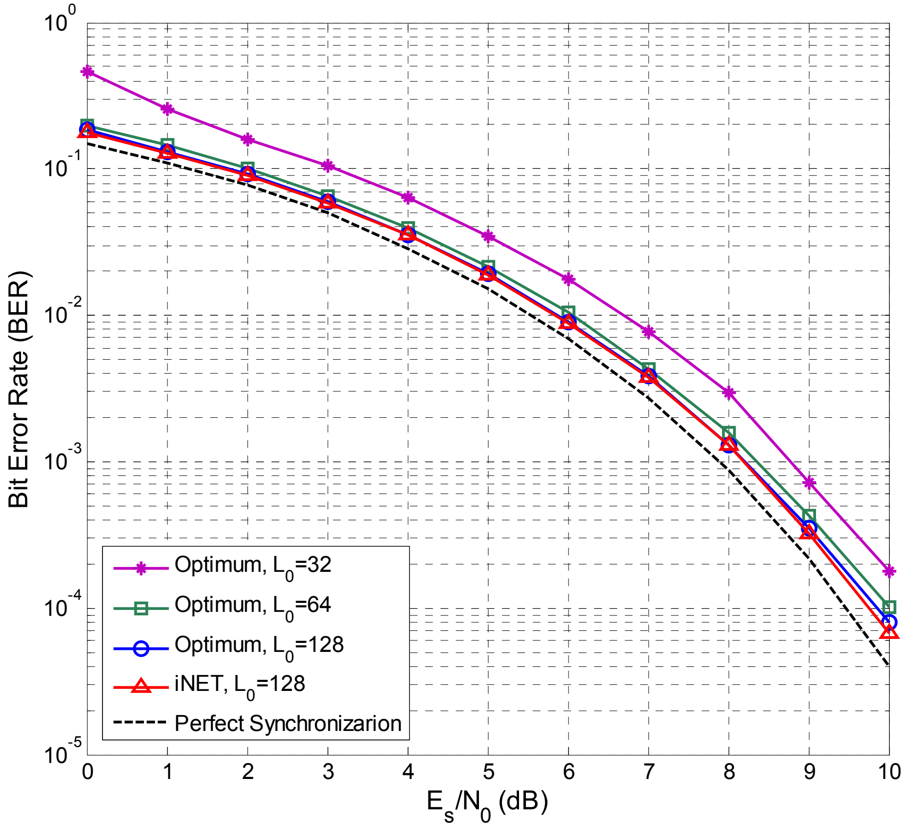
- SOQPSK-TG's response is approximated by SOQPSK-MIL's
- Derivation of joint ML estimation algorithm is similar to the CPM's when h=1/2 and *M*=2

Estimation Error Variances for SOQPSK: Optimum vs. iNET preambles



- Performance degradation due to transitions in the iNET preamble
- Phase error variance has the same behavior as the frequency’s

Overall Performance of SOQPSK: Optimum vs. iNET preambles



Preamble	dB loss @ BER=10^{-4}
iNET	0.4
L_0 =128	0.3
L_0 = 64	0.6
L_0 = 32	1

- iNET preamble performs very close to the optimum one when L_0=128
- The optimum preamble with L_0 =64 is within 0.2 dB of the iNET's
- False locks for L_0 =32 at low SNRs

Conclusions

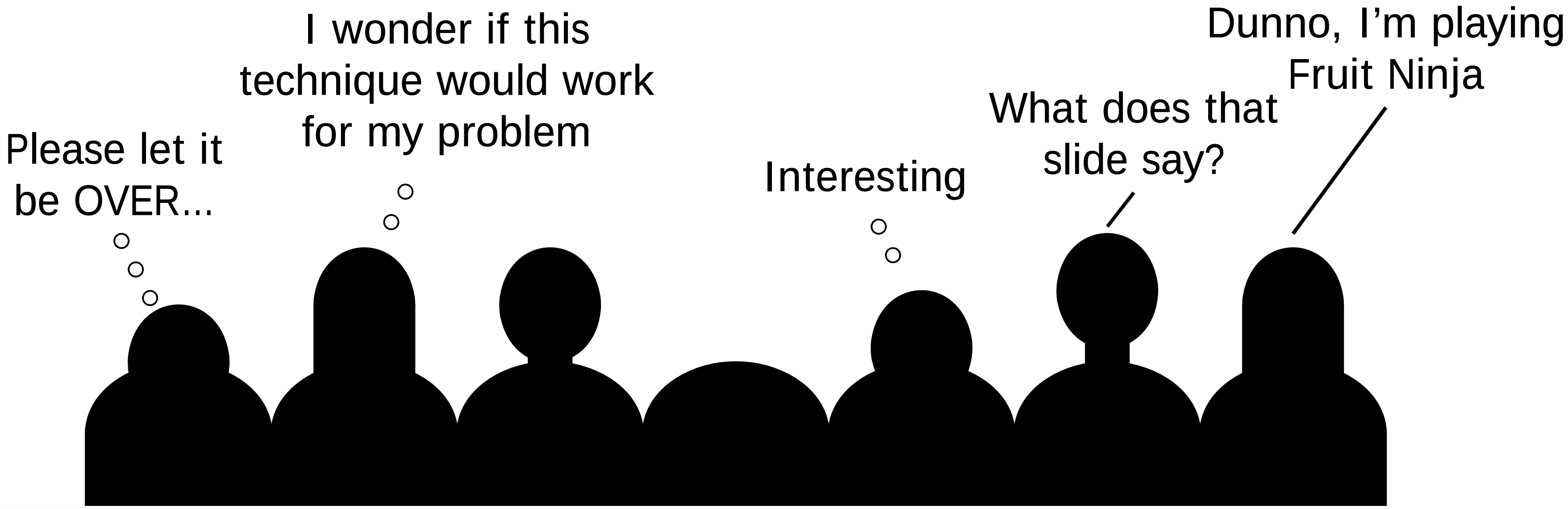
- Gave an overview of feedforwad synchronization methods for CPM
- Derived the optimum training sequence for CPM in AWGN
  - Minimizes the CRBs for all three synchronization parameters
  - Applicable to the entire CPM family including M-ary and partial response
- Developed a joint ML DA estimation algorithm for timing and carrier recovery
  - Designed based on the optimized training sequence
  - Performs within 0.5 dB of the CRB at SNRs as low as 0 dB for L_0 = 64
- Simulated a burst-mode CPM receiver: only 0.1 dB SNR loss with the optimum preamble of L_0=64
- Applied CPM results to SOQSPK for iNET standard
- **Final result**: developed a complete feedforward DA synchronization scheme for general CPM including: best training sequence, timing and carrier recovery, and frame synchronization

Areas of Future Study

- Application to frequency-selective channels:
  - Training sequence design
  - Timing and carrier recovery
  - Frame synchronization
- Joint synchronization and channel estimation
- Analysis of frame synchronization algorithm as it can be applied to other modulations,
  - Probability of false lock (SoS estimator)
  - Probabilities of false alarm and detection (SoS detector)
- Hardware implementation and trade-offs between complexity and performance

Thank You!

Questions?



Derivation of CRB

- FIM, for CPM and AWGN, is computed via

$$\mathbf{I}_{i,j}(\mathbf{u}) = -\frac{2}{N_0}\int_0^{T_0} \mathrm{Re}\left[s(t,\mathbf{u},\boldsymbol{\alpha}) \frac{\partial^2 s^*(t,\mathbf{u},\boldsymbol{\alpha})}{\partial u_i \partial u_j} \right] dt$$

$$\underbrace{s(t,\mathbf{u},\boldsymbol{\alpha}) = \sqrt{\frac{E_s}{T_s}} e^{j(2\pi f_d t + \theta)} e^{j\phi(t-\tau;\boldsymbol{\alpha})}}_{\text{Received Preamble w/o noise}}$$

$$\underbrace{T_0 = L_0 T_s}_{\text{Preamble Duration}}$$

- Closed-form FIM for CPM and joint estimation:

$$\mathbf{I}(\mathbf{u}) = \frac{1}{T_s}\left(\frac{E_s}{N_0}\right)\begin{bmatrix} 8\pi^2 T_0^3/3 & 2\pi T_0^2 & -8\pi^2 hA \\ 2\pi T_0^2 & 2T_0 & -2\pi hB \\ -8\pi^2 hA & -2\pi hB & 8\pi^2 h^2 C \end{bmatrix}$$

$$A = \sum_{i=0}^{L_0-1} \alpha_i \int_0^{T_0} t g(t - iT_s - \tau)\, dt$$

$$B = \sum_{i=0}^{L_0-1} \alpha_i \int_0^{T_0} g(t - iT_s - \tau)\, dt$$

$$C = \sum_{i=0}^{L_0-1}\sum_{j=0}^{L_0-1} \alpha_i \alpha_j \int_0^{T_0} g(t - iT_s - \tau) g(t - jT_s - \tau)\, dt$$

$g(t)$: CPM Frequency Pulse

Training Sequence for Symbol Timing

- Symbol Timing CRB should be *minimized*,

$$\mathrm{CRB}(\tau\,|\,\boldsymbol{\alpha}) = \left(8\pi^2 h^2 C - \underbrace{\begin{bmatrix} -8\pi^2 hA & -2\pi hB \end{bmatrix} \begin{bmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} \\ \mathbf{I}_{21} & \mathbf{I}_{22} \end{bmatrix}^{-1} \begin{bmatrix} -8\pi^2 hA \\ -2\pi hB \end{bmatrix}}_{\text{Quadratic form}:\,\mathbf{v}^{\mathrm{T}}\mathbf{J}\mathbf{v} \geq 0} \right)^{-1}$$

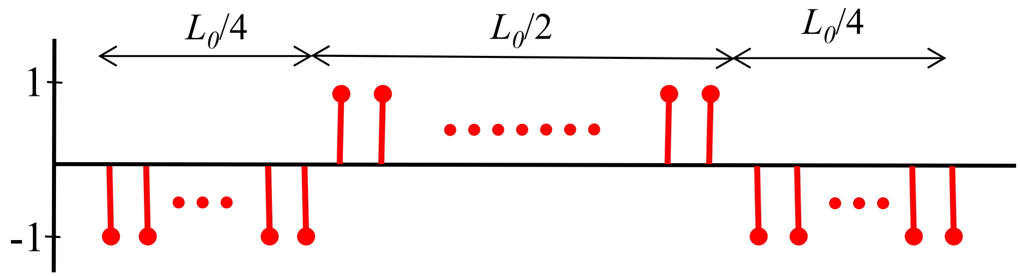
- The CRB's denominator should be *maximized*, or, simultaneously
  1. Maximize C
  2. Minimize the quadratic form: $\mathbf{J}$ is positive-definite $\Rightarrow \mathbf{v}^T\mathbf{J}\mathbf{v} \geq 0 \Rightarrow$ It is minimized for $\mathbf{v} = \mathbf{0} \Rightarrow A = B = 0$
- Assuming C is <u>independent</u> of (A,B): $\boldsymbol{\alpha}_{\text{opt}} = \arg\max_{\boldsymbol{\alpha}} C$ subject to $A = B = 0$
- In practice, $C \approx R_g(0)\sum_{i=0}^{L_0-1}\alpha_i^2 + 2R_g(T_s)\sum_{i=0}^{L_0-1}\alpha_i a_{i+1}$
  1. Full-response CPM: $R_g(T_s) = 0 \Rightarrow C = \text{const.} \Rightarrow A = B = 0$ is sufficient.
  2. Partial-response CPM: $R_g(T_s) > 0 \Rightarrow \text{maximize} \sum_{i=0}^{L_0-2}\alpha_i\alpha_{i+1}$ AND $A = B = 0$.

Proposing the training sequence

- A=B=0 is satisfied when:

$$A' = \sum_{i=1}^{L_0} i\alpha_{i-1} = 0 \text{ AND } B' = \sum_{i=1}^{L_0} \alpha_{i-1} = 0$$

- C is maximized when $\sum_{i=0}^{L_0-2} \alpha_i \alpha_{i+1}$ is maximized or *sign transitions* are minimized in the sequence.
- We propose the following binary sequence:



- It satisfies A'=B'=0,
  1. Optimum sequence for full-response CPM.
  2. Only 2 transitions => *Near-optimum* sequence for partial-response CPM.

Generalization of the Proposed Sequence

- We can reduce the approximations for partial-response by using $L_1 = L_0 - \lfloor L/2 \rfloor$ rather than L_0 to reduce the *edge-effect* (EF)
- *M*-ary CPM: $C \propto \alpha_i^2 \Rightarrow \text{Set } \alpha_i\text{'s to maximum value in the constelation}$
- Frequency and Phase CRBs:

$$\mathrm{CRB}(f_d \mid \boldsymbol{\alpha}) = \frac{3}{2\pi^2 L_0^3} \times \frac{CL_0 - B^2/4}{CL_0 - B^2/4 - \underbrace{\frac{3}{4L_0^2}(BL_0 - 4A)^2}_{\lambda_2}} \geq \frac{3}{2\pi^2 L_0^3} \qquad \mathrm{CRB}(\theta \mid \boldsymbol{\alpha}) = \frac{2}{L_0} \times \frac{CL_0^3 - 3A^2}{CL_0^3 - 3A^2 - \underbrace{(BL_0 - 3A)^2}_{\lambda_1}} \geq \frac{2}{L_0}$$

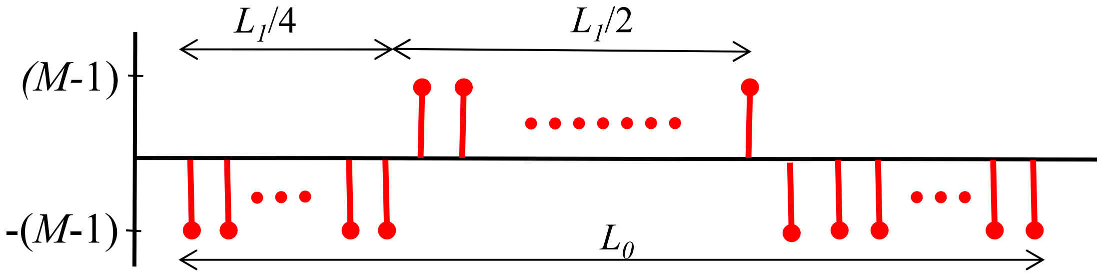
  - ❖ Equalities hold, if $\lambda_1 = \lambda_2 = 0$ or $A = B = 0$ → *Proposed sequence* for symbol timing.

- Approximations:

	Frequency	Phase	Timing
Full-response	Exact	Exact	Exact
Partial-response	EF	EF	C and EF

Proposed Training Sequence and its Approximations

- We propose the following sequence for *general* CPM and *all* synchronization parameters:



- EF is reduced as the sequence length is increased
- Approximations on C for partial-response CPM and symbol timing:

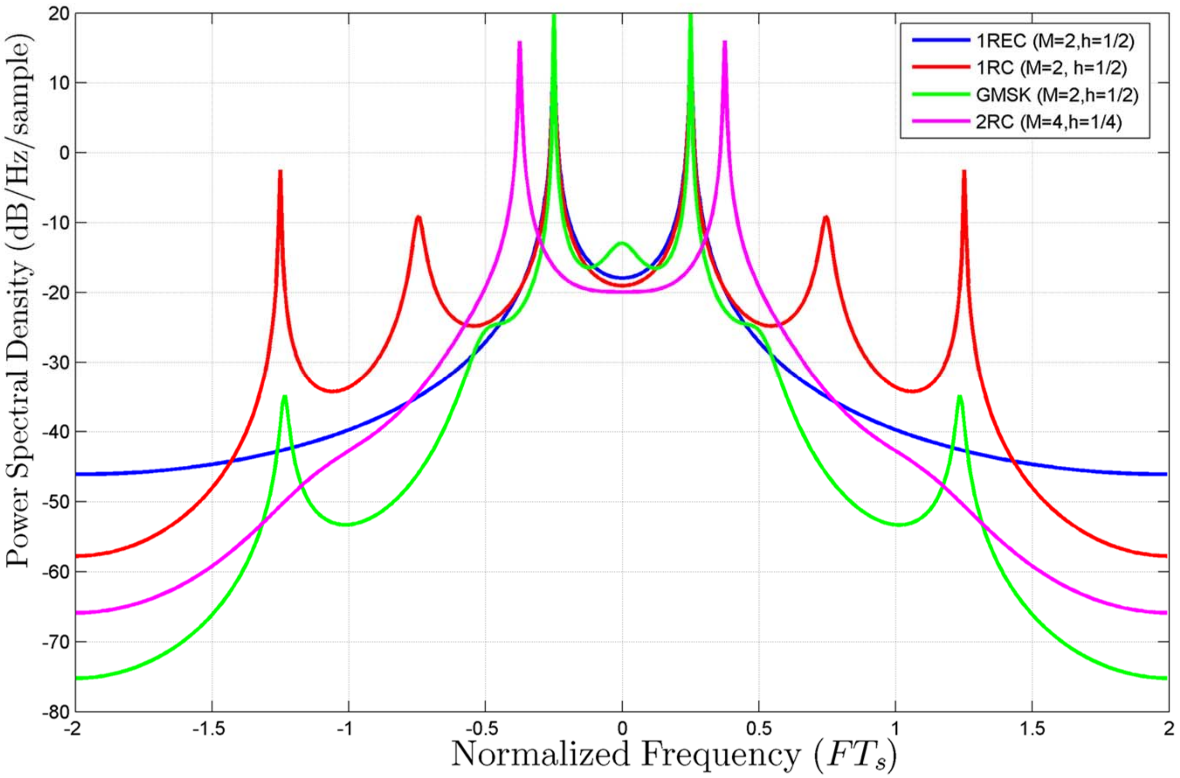
$$\frac{\mathrm{CRB}(\tilde{\boldsymbol{\alpha}} \mid \tau)}{\mathrm{CRB}(\boldsymbol{\alpha}^{*} \mid \tau)} = \frac{\tilde{C} - 8R_g(T_s)}{\tilde{C}} = 1 - \frac{8R_g(T_s)}{L_0 R_g(0) + 2(L_0 - 1)R_g(T_s)} \xrightarrow[L_0 \to \infty]{} 1$$

$\tilde{\boldsymbol{\alpha}}$: Hypothetical sequence

$\boldsymbol{\alpha}^{*}$: Proposed sequence

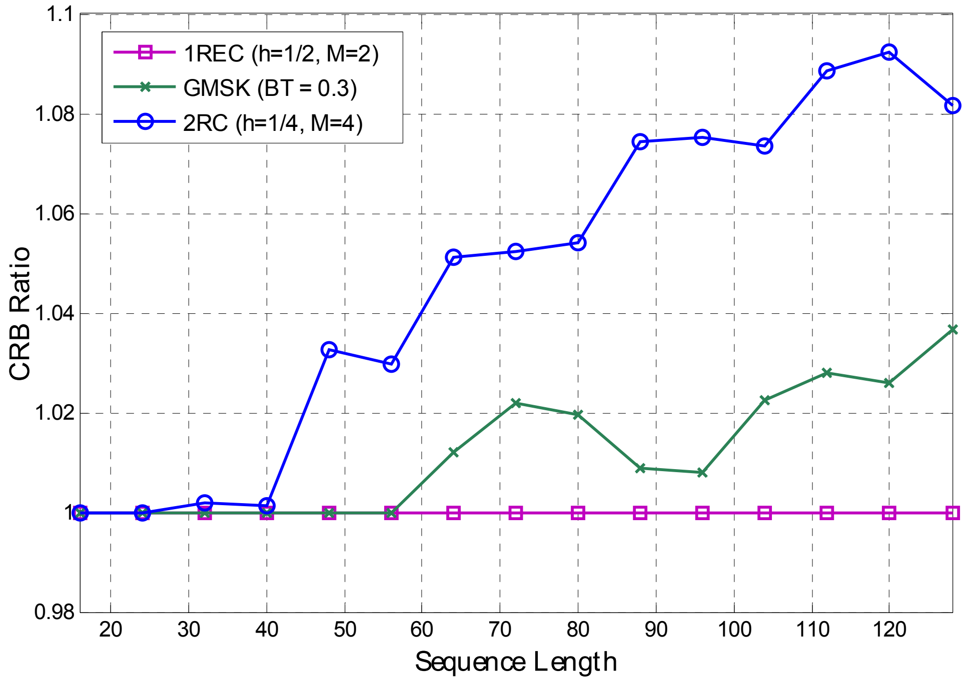
- The proposed sequence is *asymptotically* optimum

PSD of the Optimum Sequence



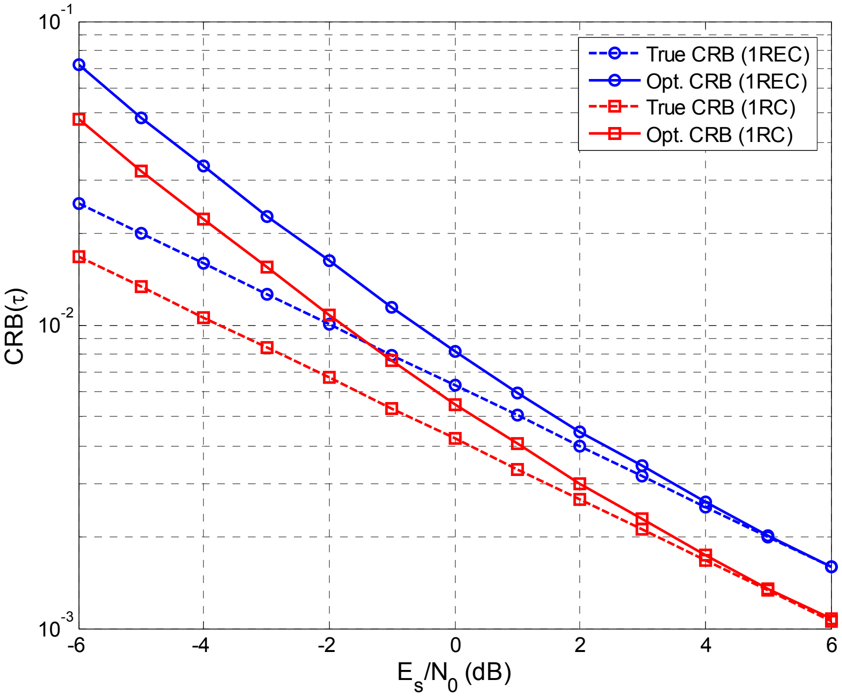
Computer Search for the Optimum Sequence: Genetic Algorithm (GA)

- The ratio of the minimum CRB found via GA to the proposed sequence CRB:



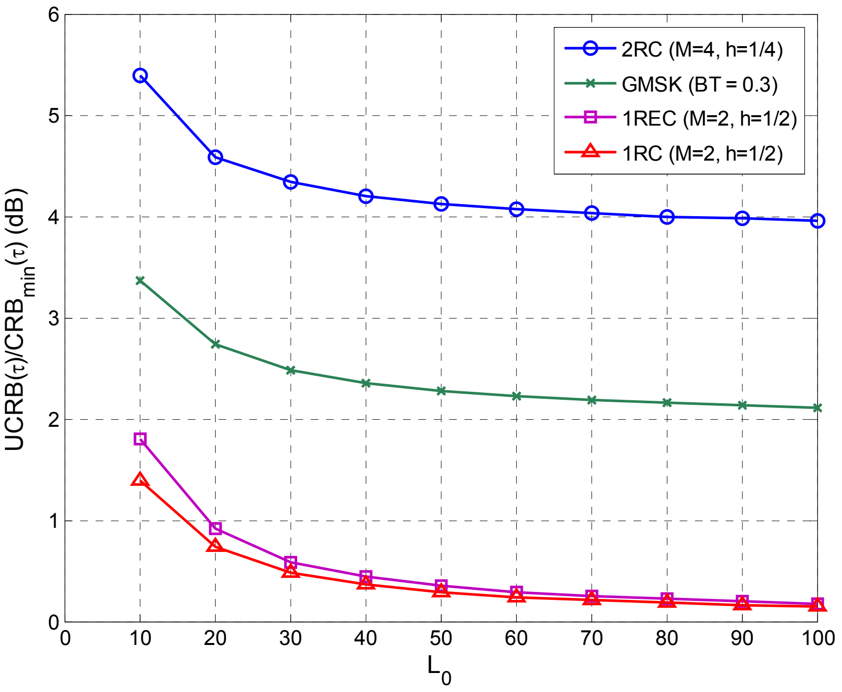
- The GA is unable to find a better sequence than the proposed one even at short to moderate lengths

Random vs. Optimum Sequences: Symbol Timing Estimation



True CRB vs. Opt. CRB for L=32

Comparison of estimator performance for "unknown and random" with "known and optimum" training sequences.

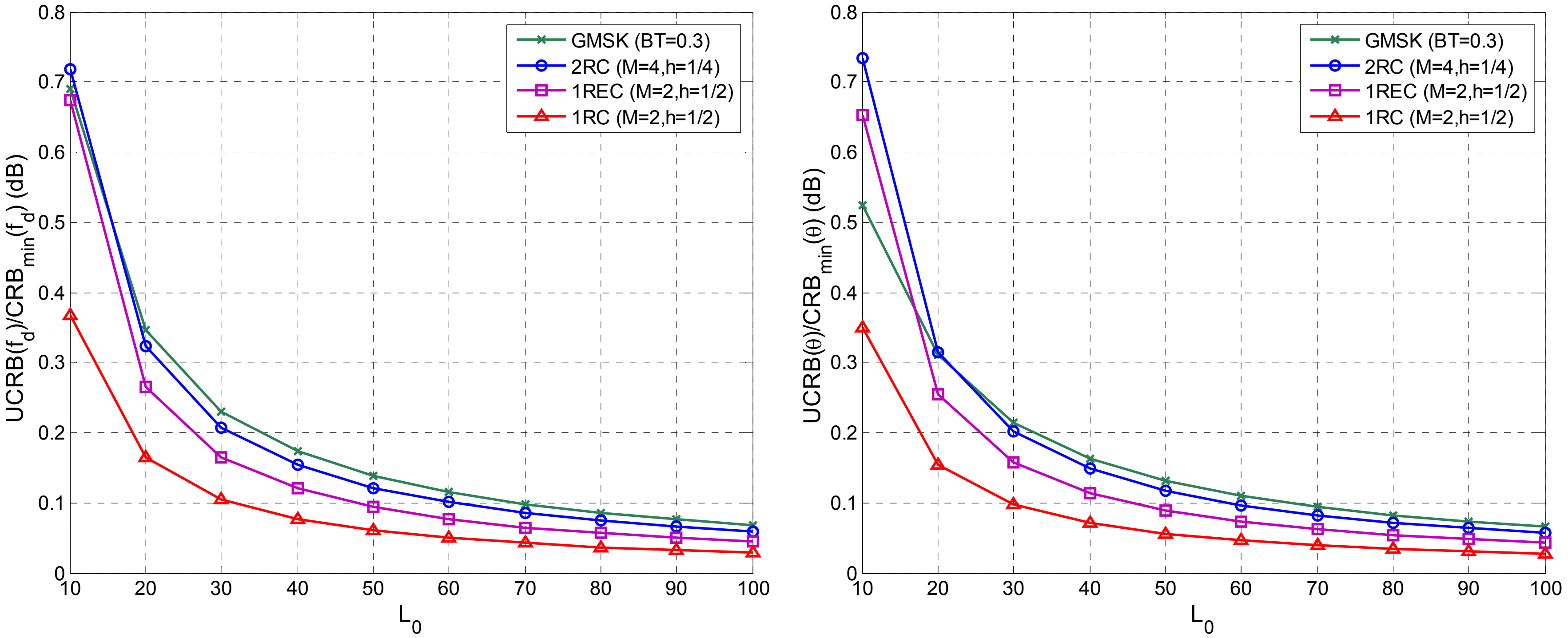


UCRB vs. Opt. CRB

Comparison of estimator performance for "known but randomly selected" with "known and optimum" training sequences.

Random vs. Optimum Sequences: Frequency and Phase Estimation

- UCRB vs. Optimum CRB



- Frequency and phase estimation are less sensitive to the selection of training sequence

Derivation of CRB

- Assumption: Joint estimation of $\mathbf{u} = [f_d\, , \theta, \tau]^{\mathrm{T}}$ in AWGN,

$$r(t) = \sqrt{\frac{E_s}{T_s}} e^{j(2\pi f_d + \theta)} e^{j\phi(t-\tau,\boldsymbol{\alpha})} + w(t)$$

- The FIM elements:

$$\mathbf{I}(\mathbf{u})_{i,j} = \frac{2}{N_0} \int_0^{T_0} \mathrm{Re}\left[\frac{\partial s(t,\mathbf{u})}{\partial u_i} \frac{\partial s^*(t,\mathbf{u})}{\partial u_j} \right] dt$$

  - Observation interval: $T_0 = L_0 T_s$ or L_0 symbols
  - Computation of three partial derivatives:

$$\frac{\partial s(t,\mathbf{u})}{\partial f_d} = j2\pi t s(t,\mathbf{u}), \quad \frac{\partial s(t,\mathbf{u})}{\partial \theta} = js(t,\mathbf{u}), \quad \frac{\partial s(t,\mathbf{u})}{\partial \tau} = j\frac{\partial \phi(t-\tau;\boldsymbol{\alpha})}{\partial \tau} s(t,\mathbf{u})$$

  - Note for CPM: $s(t,\mathbf{u})s^*(t,\mathbf{u}) = 1$
  - Frequency pulse: $\frac{\partial \phi(t-\tau;\boldsymbol{\alpha})}{\partial \tau} = -\sum_{i=0}^{L_0-1} \pi \alpha_i g(t - iT_s - \tau)$ where $g(t) = \partial q(t)/\partial t \rightarrow$ Frequency Pulse

Derivation of CRB
More Simplifications

- A, B, and C relate CRBs to frequency pulse and data symbols. They are approximated as,

$$A \approx \sum_{i=0}^{L_0-1} \alpha_i \left(\Gamma + i \frac{T_s}{2} \right) \text{ where } \Gamma = \int_0^{LT_s} ug(u)du \qquad B \approx \frac{1}{2} \sum_{i=0}^{L_0-1} \alpha_i$$

$$C \approx \sum_{i=0}^{L_0-1} \sum_{j=0}^{L_0-1} \alpha_i \alpha_j R_g\left((i-j)T_s\right) \text{ where } R_g(t) = \int_{-\infty}^{\infty} g(u)g(u+t)du$$

- Edge-effect: $L_1 = L_0 - \lfloor L/2 \rfloor$ for partial-response CPM
- Correlation function of $g(t)$:

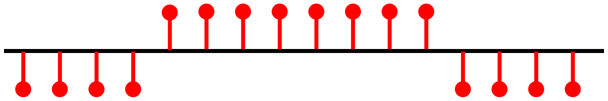
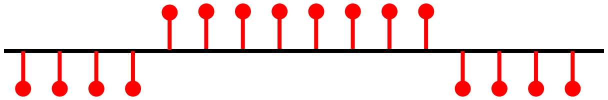
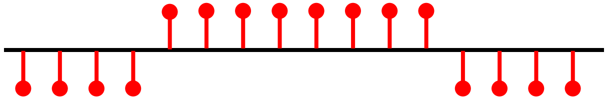
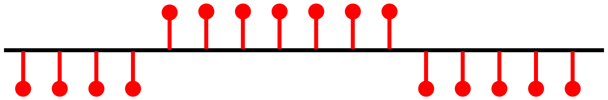
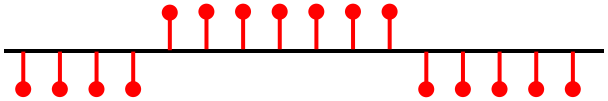
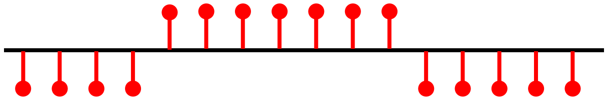
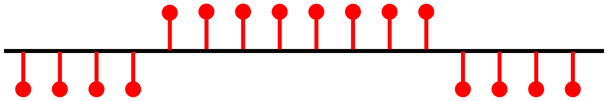
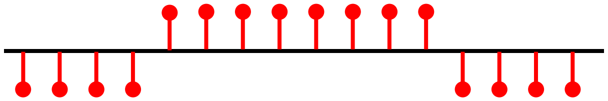
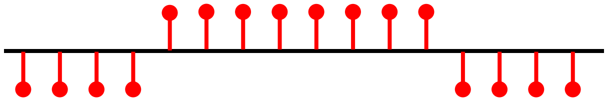
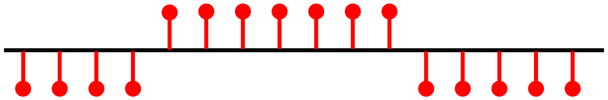
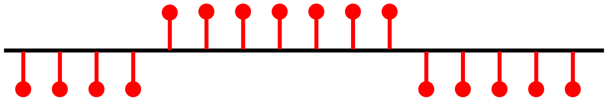
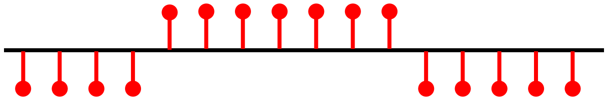
$$R_g(nT_s) \approx 0 \text{ for } n > 1 \Rightarrow C \approx R_g(0) \sum_{i=0}^{L_0-1} \alpha_i^2 + 2R_g(T_s) \sum_{i=0}^{L_0-1} \alpha_i a_{i+1}$$

Pulse Shape	**2REC**	**3REC**	**2RC**	**3RC**	**GMSK**
$R_g(0)$	0.125	0.0833	0.1875	0.125	0.1327
$R_g(\pm T_s)$	0.0625	0.0556	0.0312	0.0589	0.0551
$R_g(\pm 2T_s)$	0	0.0278	0	0.0036	0.0036

≈ 0

Exhaustive Computer Search for CPM

- Exact computer exhaustive search vs. approximated proposed sequence:

CPM	Exhaustive Search	Proposed Sequence	ΔCRB
1REC			0
2REC			0.6%
3REC			0
1RC			0
2RC			0.7%
3RC			0

Length = 16, Binary CPM

CRB for Random Sequences

- Comparison of the optimum sequence and a random sequence
- True CRB:
  1. The likelihood function is averaged over $\boldsymbol{\alpha}$
  2. Only feasible for MSK-type CPMs, i.e. binary, $h=1/2$, and full-response

$$\mathbf{I}(\mathbf{u})_{k,l} = \left(\frac{2E_s}{N_0}\right)^2 \sum_{i=0}^{L_0-1}\sum_{j=0}^{L_0-1} E\left\{\frac{\partial \hat{a}_i}{\partial u_k}\frac{\partial \hat{a}_j}{\partial u_l}\tanh\left(\frac{2E_s}{N_0}\hat{a}_i\right)\tanh\left(\frac{2E_s}{N_0}\hat{a}_j\right)\right\}$$

  3. The above expectation is computed via simulations

- Unconditional CRB (UCRB):
  1. Conditional CRB is averaged over $\boldsymbol{\alpha}$.

$$E_{\mathbf{r}|\boldsymbol{\alpha}}\left[(\hat{\tau}-\tau)^2\right] \geq \mathrm{CRB}(\tau\mid\boldsymbol{\alpha}) \xrightarrow{\text{Averaging over }\boldsymbol{\alpha}} E_{\mathbf{r}}\left[(\hat{\tau}-\tau)^2\right] \geq E_{\boldsymbol{\alpha}}\left[\mathrm{CRB}(\tau\mid\boldsymbol{\alpha})\right] = \mathrm{UCRB}(\tau)$$

  2. A closed-form expression was found for symbol timing

CRB for Random Data Sequence:
True CRB

- NDA estimators with random (unknown) data sequence:
  - Computation of *true* CRB,   as a nuisance parameter.
  - Quite complex; may not be feasible for general CPM.
- Special case: binary, h=1/2 and full-response:
  1. PAM representation:

$$s(t)=\sum_i a_i c_0\left(t-iT_s\right)\xrightarrow{\text{Receiver}}\tilde{a}_n=\begin{cases}\int_{2T_s}\mathrm{Re}\left[e^{-j(2\pi f_d t+\theta)}r(t)c_0\left(t-nT_s-\tau\right)\right]dt & n-\text{even}\\ \int_{2Ts}\mathrm{Im}\left[e^{-j(2\pi f_d t+\theta)}r(t)c_0\left(t-nT_s-\tau\right)\right]dt & n-\text{odd}\end{cases}$$

  2. a_i's *pseudo symbols*: independent Gaussian RVs.
  3. Averaging LF over a_i's + partial derivatives→ FIM elements:

$$\mathbf{I}(\mathbf{u})_{k,l}=\left(\frac{2E_s}{N_0}\right)^2\sum_{i=0}^{L_0-1}\sum_{j=0}^{L_0-1}E\left\{\frac{\partial\tilde{a}_i}{\partial u_k}\frac{\partial\tilde{a}_j}{\partial u_l}\tanh\left(\frac{2E_s}{N_0}\tilde{a}_i\right)\tanh\left(\frac{2E_s}{N_0}\tilde{a}_j\right)\right\}$$

  4. *Monte-Carlo* simulations → CRBs.

CRB for Random Data Sequence: **Unconditional CRB (UCRB)**

- Averaging the conditional CRB w.r.t the training sequence results in ***unconditional*** lower bound on the MSE:

$$E_{\mathbf{r}|\boldsymbol{\alpha}}\left[(\hat{\tau}-\tau)^2\right] \geq \mathrm{CRB}(\tau \mid \boldsymbol{\alpha}) \xrightarrow{\text{Averaging over } \boldsymbol{\alpha}} E_{\mathbf{r}}\left[(\hat{\tau}-\tau)^2\right] \geq E_{\boldsymbol{\alpha}}\left[\mathrm{CRB}(\tau \mid \boldsymbol{\alpha})\right] = \mathrm{UCRB}(\tau)$$

- Symbol timing CRB can be expressed as:

$$\mathrm{CRB}(\tau|\boldsymbol{\alpha}) = \frac{T_s}{E_s/N_0}\frac{1}{\boldsymbol{\alpha}^T\mathbf{G}\boldsymbol{\alpha} - \boldsymbol{\alpha}^T\mathbf{D}\mathbf{J}\mathbf{D}^T\boldsymbol{\alpha}} = \frac{T_s}{E_s/N_0}\frac{1}{\boldsymbol{\alpha}^T\mathbf{Q}\boldsymbol{\alpha}}$$

- Where:

$$\mathbf{G} = 8\pi^2h^2\begin{bmatrix} R_g(0) & R_g(T_s) & \dots & R_g((L_0-1)T_s) \\ R_g(-T_s) & R_g(0) & R_g(T_s) & \dots \\ \vdots & \vdots & \ddots & \dots \\ R_g((1-L_0)T_s) & \dots & R_g(-T_s) & R_g(0) \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} -8\pi^2h(\Gamma+\tau/2) & -\pi h \\ -8\pi^2h(\Gamma+\tau/2+T_s/2) & -\pi h \\ \vdots & \vdots \\ -8\pi^2h(\Gamma+\tau/2+(L_0-1)T_s/2) & -\pi h \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} \\ \mathbf{I}_{21} & \mathbf{I}_{22} \end{bmatrix}^{-1}$$

- Thus:

$$\mathrm{UCRB}(\tau \mid \boldsymbol{\alpha}) = \frac{T_s}{E_s/N_0} E_{\boldsymbol{\alpha}}\left[\frac{1}{\boldsymbol{\alpha}^T\mathbf{Q}\boldsymbol{\alpha}}\right] \rightarrow \begin{cases} \text{Numerical} \\ \text{Simplify} \end{cases}$$

Computing Timing UCRB

- Define: $Z = \boldsymbol{\alpha}^T \mathbf{Q} \boldsymbol{\alpha}$
- Approximation via Taylor series:

$$\mathrm{UCRB}(\tau) = \frac{T_s}{E_s / N_0} E\left[\frac{1}{Z}\right] \approx \frac{T_s}{E_s / N_0} \frac{E\left[Z^2\right]}{E\left[Z\right]^3}$$

- Eigen-decomposition of $\mathbf{Q}$:

$$Z = \boldsymbol{\alpha}^T (\mathbf{V}^T \boldsymbol{\Lambda} \mathbf{V}) \boldsymbol{\alpha} = \mathbf{x}^{\mathrm{T}} \boldsymbol{\Lambda} \mathbf{x} = \sum_{i=1}^{L_0} \lambda_i X_i^2 \qquad \mathbf{x} = \mathbf{V}^T \boldsymbol{\alpha} = [X_1, X_2, \cdots, X_{L_0}]^T$$

- Expected value of Z

$$E\{Z\} = \sum_{i=1}^{L_0} \lambda_i E\{X_i^2\} = \sum_{i=1}^{L_0} \lambda_i \mathbf{v}_{\mathrm{i}}{}^T E\{\boldsymbol{\alpha}\boldsymbol{\alpha}^T\} \mathbf{v}_{\mathrm{i}} = \frac{M^2 - 1}{3} \sum_{i=1}^{L_0} \lambda_i$$

- Using the same approach and employing central limit theorem:

$$E\{Z^2\} \approx \frac{(M^2-1)^2}{9} \left[\sum_i \lambda_i\right]^2 + \frac{2(M^2-1)^2}{9} \sum_i \lambda_i^2 = \frac{(M^2-1)^2}{9} \mathrm{trace}(\mathbf{Q})^2 + \frac{2(M^2-1)^2}{9} \mathrm{trace}(\mathbf{Q}^2)$$

UCRB: Computation of $E\{Z^2\}$

- Recall $Z = \sum_{i=1}^{L_0} X_i^2$, the second moment of Z is written as

$$E\{Z^2\} = \boldsymbol{\lambda}^T E\{\mathbf{x_2}\mathbf{x_2^T}\}\boldsymbol{\lambda} = \boldsymbol{\lambda}^T \mathbf{C}_{\mathbf{x_2}}\boldsymbol{\lambda}$$

- Where $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_{L_0}]^T$ and $\mathbf{x_2} = [X_1^2, X_2^2, \ldots, X_{L_0}^2]^T$
- $\mathbf{C}_{\mathbf{x_2}}$ requires computation of $E\{X_i^4\}$ and $E\{X_i^2 X_j^2\}$.
- Approximation:

$$X_i = v_1^{(i)}\alpha_1 + v_2^{(i)}\alpha_1 + \cdots + v_{L_0}^{(i)}\alpha_{L_0}$$

$$\text{Centrallimit theorem} \rightarrow X_i \sim N\left(0, \frac{M^2-1}{3}\right)$$

- X_i`s are uncorrelated and hence independent.
- Finally

$$[\mathbf{C}_{\mathbf{x_2}}]_{ij} \approx \begin{cases} \dfrac{(M^2-1)^2}{3} & i = j \\ \dfrac{(M^2-1)^2}{9} & i \neq j \end{cases}$$

Maximization of LLF (Details 1/2)

1. Discrete-time LLF:

$$\Lambda(\mathbf{r};\nu,\theta,\varepsilon) \approx \mathrm{Re}\left[\sum_{n=0}^{NL_0-1} e^{-j(2\pi n\nu+\theta)} r[n] s_\varepsilon^*[n]\right]$$

2. Approximated CPM signal:

$$s_\varepsilon[n] \approx \begin{cases} \exp[-j(M-1)\pi h(n/N-\varepsilon)] & n \le n < NL_0/4 \\ \exp[+j(M-1)\pi h(n/N-L_0/2-\varepsilon)] & NL_0/4 \le n < 3NL_0/4 \\ \exp[-j(M-1)\pi h(n/N-L_0-\varepsilon)] & 3NL_0/4 \le n < NL_0 \end{cases}$$

3. Approximated LLF

$$\Lambda(\mathrm{r};\nu,\theta,\varepsilon) \approx \mathrm{Re}\left\{ e^{-j\theta}\left[\sum_{n=0}^{NL_0/4-1} e^{-j2\pi\upsilon n} r[n] e^{j(M-1)\pi h(n/N-\varepsilon)} + \sum_{n=NL_0/4}^{3NL_0/4-1} e^{-j2\pi\upsilon n} r[n] e^{-j(M-1)\pi h(n/N-L_0/2-\varepsilon)} \right.\right.$$

$$\left.\left. + \sum_{n=3NL_0/4}^{NL_0-1} e^{-j2\pi\upsilon n} r[n] e^{j(M-1)\pi h(n/N-L_0-\varepsilon)} \right]\right\}$$

Maximization of LLF (Details 2/2)

4. Rearrange LLF as

$$\Lambda(\mathbf{r};\nu,\theta,\varepsilon) \approx \mathrm{Re}\left\{ e^{-j\theta}\left[e^{-j(M-1)\pi h\varepsilon}\lambda_1(\nu) + e^{j(M-1)\pi h\varepsilon}\lambda_2(\nu)\right]\right\}$$

5. If $\Gamma(\nu,\varepsilon) = e^{-j(M-1)\pi h\varepsilon}\lambda_1(\nu) + e^{j(M-1)\pi h\varepsilon}\lambda_2(\nu)$, the LLF is maximized when phase is

$$\tilde{\theta} = \arg\{\Gamma(\nu,\varepsilon)\}$$

6. The LLF becomes $\left|\Gamma(\tilde{\nu},\tilde{\varepsilon})\right|$, which is maximized by maximizing

$$\left|\Gamma(\tilde{\nu},\tilde{\varepsilon})\right|^2 = \left|\lambda_1(\tilde{\nu})\right|^2 + \left|\lambda_2(\tilde{\nu})\right|^2 + 2\,\mathrm{Re}\left[e^{-j2(M-1)\pi h\tilde{\varepsilon}}\lambda_1(\tilde{\nu})\lambda_2^*(\tilde{\nu})\right]$$

7. That is,

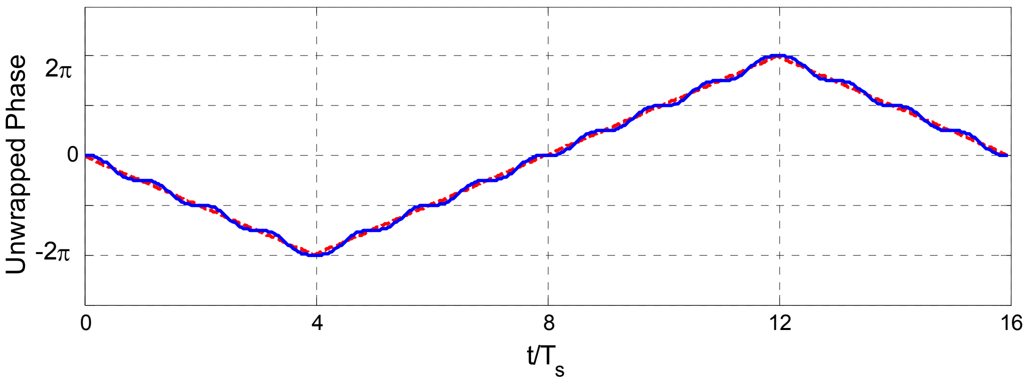
$$\tilde{\varepsilon} = \frac{\arg\{\lambda_1(\tilde{\nu})\lambda_2^*(\tilde{\nu})\}}{2(M-1)\pi h}$$

$$\hat{\nu} = \arg\max_{\tilde{\nu}}\left\{\left|\lambda_1(\tilde{\nu})\right| + \left|\lambda_2(\tilde{\nu})\right|\right\}$$

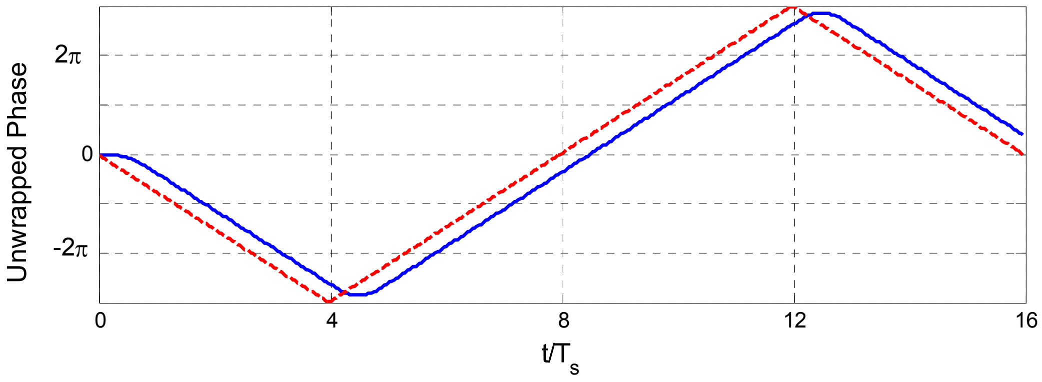
■

Linear Phase Approximation: Other CPM Examples

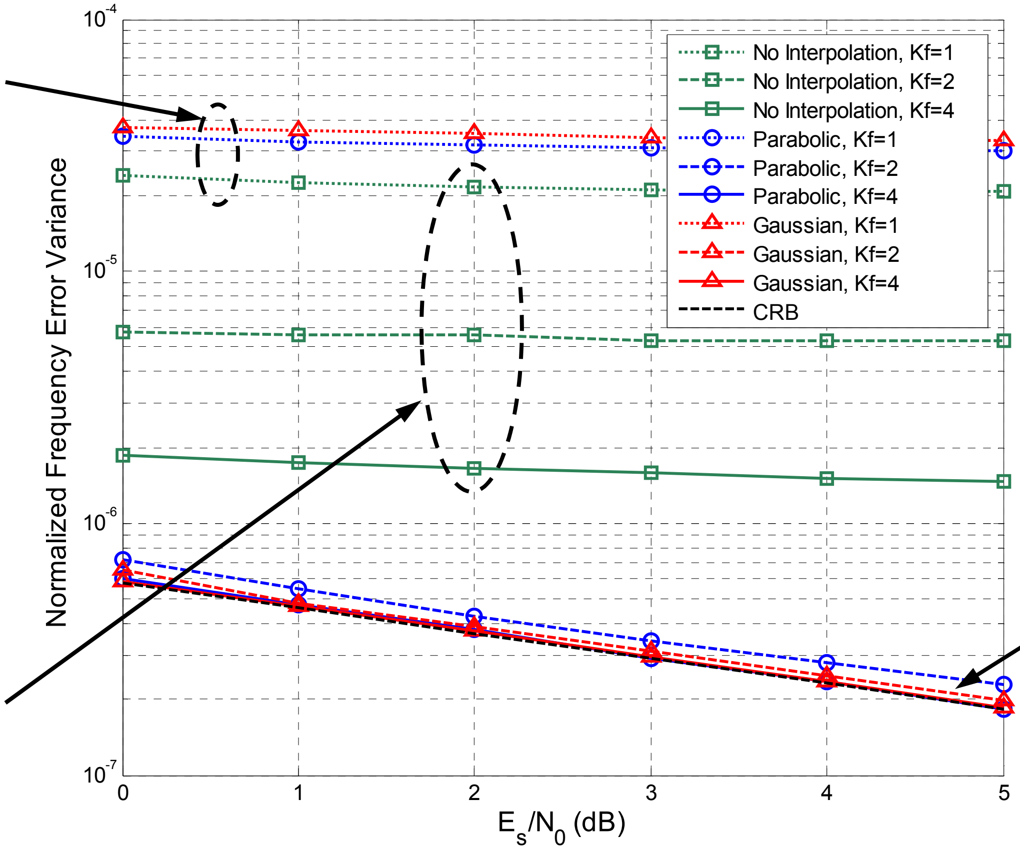
1RC (M=2,h=1/2)



2RC (M=4,h=1/4)



FFT and Interpolation Precision



No zero padding

No interpolation

Our choice:
K_f=2, Gaussian

- Frequency MSE for different interpolators and zero padding factors.
- Both interpolation and zero padding must be present based on simulations.

SoS Estimation Algorithm (Details 1/3)

1. Likelihood function of all unknown parameters

$$p(\mathbf{r};\delta,\nu,\theta,\boldsymbol{\alpha}_d)=\frac{1}{(\pi\sigma^2)^{N_w}}\exp\left(-\frac{1}{\sigma^2}\sum_{n=0}^{\delta-1}\left|r[n]\right|^2\right)\exp\left(-\frac{1}{\sigma^2}\sum_{n=\delta}^{N_w-1}\left|r[n]-s[n-\delta]e^{j(2\pi\nu n+\theta)}\right|^2\right)$$

2. Drop constant factors

$$p(\mathbf{r};\delta,\nu,\theta,\boldsymbol{\alpha}_d)=\exp\left(\frac{\delta-N_w}{\sigma^2}\right)\exp\left(\frac{2}{\sigma^2}\sum_{n=\delta}^{N_w-1}\mathrm{Re}\left\{r^*[n]s[n-\delta]e^{j(2\pi\nu n+\theta)}\right\}\right)$$

3. Approximate the exponential function using 2nd degree Taylor series

$$\begin{aligned}p(\mathbf{r};\delta,\nu,\theta,\boldsymbol{\alpha}_d)\approx C(\delta)\Bigg(&1+\frac{2}{\sigma^2}\sum_{n=\delta}^{N_w-1}\mathrm{Re}\left\{r^*[n]s[n-\delta]e^{j(2\pi\nu n+\theta)}\right\}\\&+\frac{1}{\sigma^4}\sum_{n=\delta}^{N_w-1}\sum_{m=\delta}^{N_w-1}\mathrm{Re}\left\{r^*[n]r^*[m]s[n-\delta]s[m-\delta]e^{j(2\pi\nu(m+n)+2\theta)}\right\}\\&+\frac{1}{\sigma^4}\sum_{n=\delta}^{N_w-1}\sum_{m=\delta}^{N_w-1}\mathrm{Re}\left\{r^*[n]r[m]s[n-\delta]s[m-\delta]e^{j(2\pi\nu(n-m)+2\theta)}\right\}\end{aligned}$$

SoS Estimation Algorithm (Details 2/3)

4. Average the likelihood function w.r.t θ

$$p(\mathbf{r};\delta,\nu,\boldsymbol{\alpha}_d)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Lambda(r;\delta,\nu,\theta,\alpha_d)d\theta$$

$$\approx C(\delta)\frac{1}{\sigma^4}\sum_{n=\delta}^{N_w-1}\sum_{m=\delta}^{N_w-1}\mathrm{Re}\left\{r^*[n]r[m]s[n-\delta]s^*[m-\delta]e^{j2\pi\nu(n-m)}\right\}$$

5. Rearrange the above likelihood function

$$p(\mathbf{r};\delta,\nu,\boldsymbol{\alpha}_d)\approx C(\delta)\left(\sum_{n=\delta}^{N_w-1}|r[n]|^2+2\sum_{d=1}^{N_w-\delta-1}\mathrm{Re}\left\{e^{-j2\pi d\nu}\sum_{n=\delta}^{N_w-d-1}r^*[n]r[n+d]s[n-\delta]s^*[n+d-\delta]\right\}\right)$$

6. Take the expectation w.r.t data symbols

$$p(\mathbf{r};\delta,\nu)\approx C(\delta)\left(\sum_{n=\delta}^{N_w-1}|r[n]|^2+2\sum_{d=1}^{N_p-1}\mathrm{Re}\left\{e^{-j2\pi d\nu}\left(\underbrace{\sum_{n=\delta}^{N_p+\delta-d-1}r^*[n]r[n+d]s[n-\delta]s^*[n+d-\delta]}_{Preamble}+\underbrace{R_{ss}(d)\sum_{n=N_p+\delta}^{N_w-d-1}r^*[n]r[n+d]}_{Random\ Data}\right)\right\}\right)$$

SoS Estimation Algorithm (Details 3/3)

7. A frequency estimate can be derived from each term in the LLF

$$\hat{\nu}_d = \frac{1}{2\pi d}\arg\left\{ \sum_{n=\delta}^{N_p+\delta-d-1} r^*[n]r[n+d]s[n-\delta]s^*[n+d-\delta] + R_{ss}(d)\sum_{n=N_p+\delta}^{N_w-d-1} r^*[n]r[n+d]\right\}$$

8. If we use each estimate in its corresponding term, the LLF becomes independent of frequency offset

$$p(\mathbf{r};\delta) \approx C(\delta)\left(\sum_{n=\delta}^{N_w-1}|r[n]|^2 + 2\sum_{d=1}^{N_p-1}\left|\sum_{n=\delta}^{N_p+\delta-d-1} r^*[n]r[n+d]s[n-\delta]s^*[n+d-\delta] + R_{ss}(d)\sum_{n=N_p+\delta}^{N_w-d-1} r^*[n]r[n+d]\right|\right)$$

9. Since the LLF is now only a function of δ, it can be used for estimation of SoS

$$\hat{\delta} = \arg\max_{\delta}\left\{C(\delta)\left(\sum_{n=\delta}^{N_w-1}|r[n]|^2 + 2\sum_{d=1}^{N_p-1}\left|\sum_{n=\delta}^{N_p+\delta-d-1} r^*[n]r[n+d]s[n-\delta]s^*[n+d-\delta] + R_{ss}(d)\sum_{n=N_p+\delta}^{N_w-d-1} r^*[n]r[n+d]\right|\right)\right\}$$

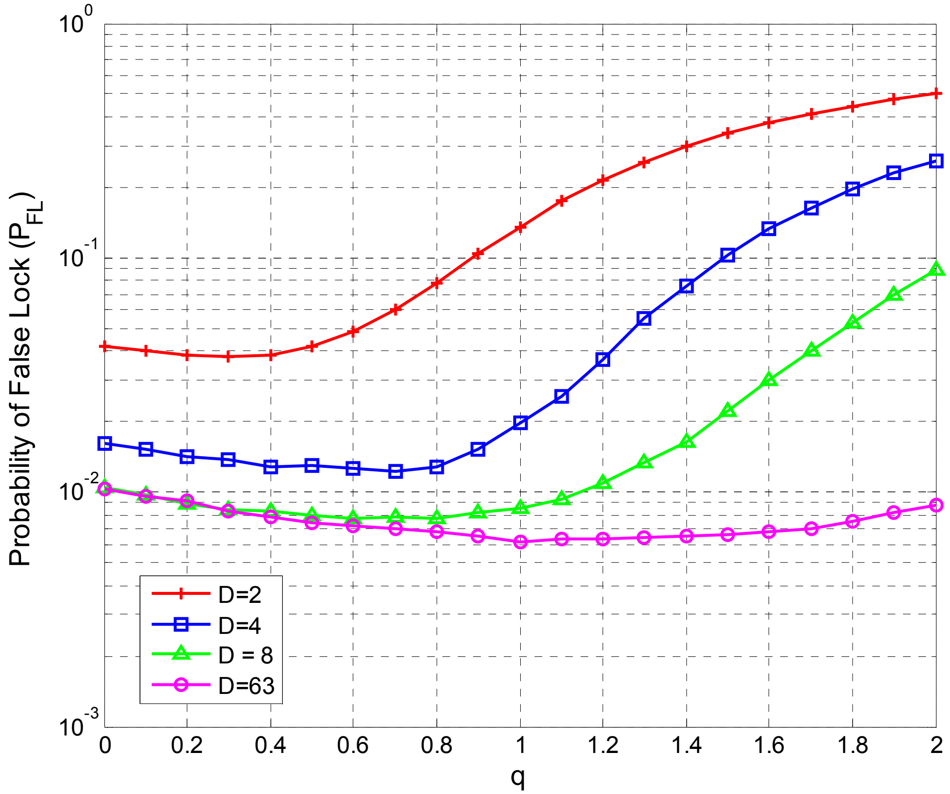
$$C(\delta) = (N_w-\delta)^q$$

■

Optimization of $C(\delta)$

GMSK
$N_p = 64$
$E_s/N_0 = 1$ dB

$$C(\delta) = (N_w - \delta)^q$$



- P_{FL} is computed via simulations
- $C(\delta)$ is more important for larger values of D