A Multipath Channel Estimation Algorithm using the Kalman Filter.

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Organization

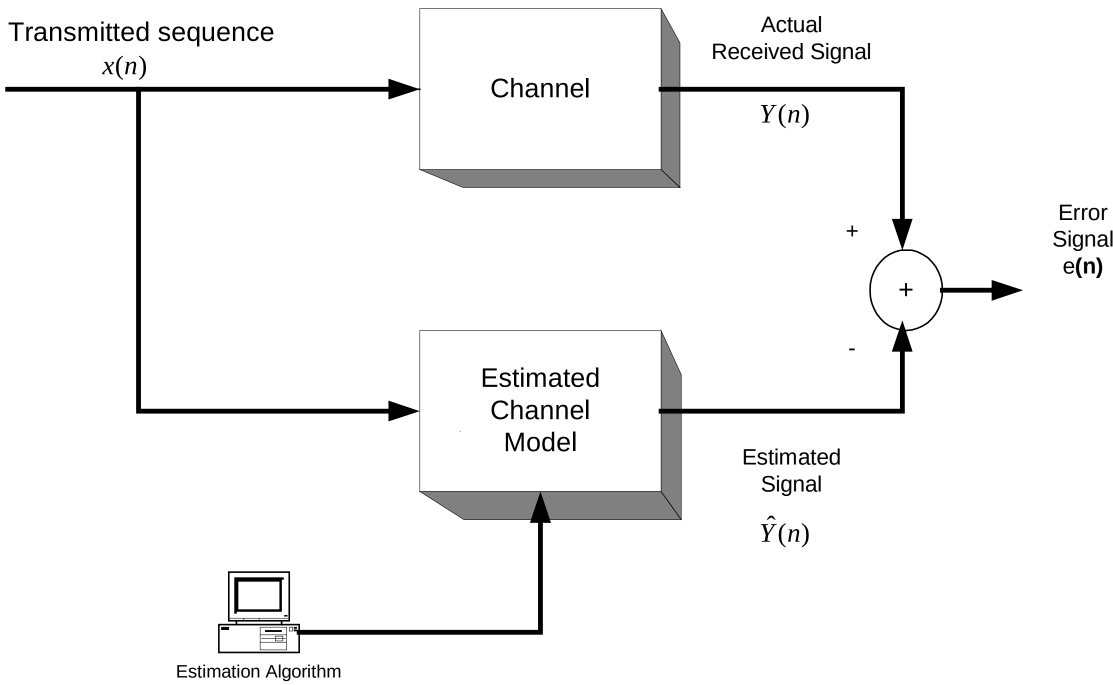
- Introduction
- Theoretical Background
- Channel Estimation Algorithm
- Conclusions
- Future Work

Introduction

Definitions:

- Channel: In its most General sense can describe everything from the source to the sink of the radio signal. Including the physical medium.
  - In this work “Channel” refers to the physical medium.
- Channel Model: Is a mathematical representation of the transfer characteristics of the physical medium.
  - Channel models are formulated by observing the characteristics of the received signal.
  - The one that best explains the received signal behavior is used to model the channel.
- Channel Estimation: The process of characterizing the effect of the physical medium on the input sequence.

General Channel Estimation Procedure



- Aim of any channel estimation procedure:
  - Minimize some sort of criteria, e.g. MSE.
  - Utilize as little computational resources as possible allowing easier implementation.
- A channel estimate is only a mathematical estimation of what is truly happening in nature.
- Why Channel Estimation?
  - Allows the receiver to approximate the effect of the channel on the signal.
  - The channel estimate is essential for removing inter symbol interference, noise rejection techniques etc.
  - Also used in diversity combining, ML detection, angle of arrival estimation etc.

Training Sequences vs. Blind Methods

- There Are two Basic types of Channel Estimation Methods:

Training Sequence methods:

- Sequences known to the receiver are embedded into the frame and sent over the channel.
- Easily applied to any communications system.
- Most popular method used today.
- Not too computationally intense.
- Has a major drawback: It is wasteful of the information bandwidth.

Blind Methods:

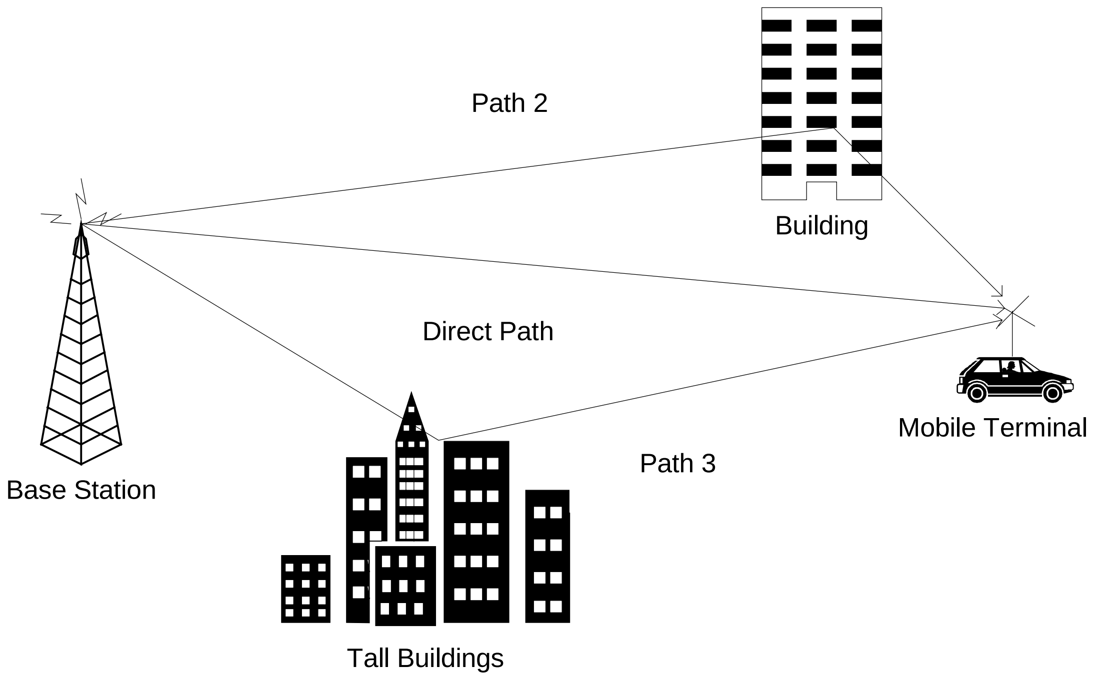
- No Training sequences required
- Uses certain underlying mathematical properties of the data being sent.
- Excellent for applications where bandwidth is scarce.
- Has the drawback of being extremely computationally intensive
- Thus hard to implement on real time systems.

Algorithm Overview

- Consider a radio communications system using training sequences to do channel estimation.
- This thesis presents a method of improving on the training sequence based estimate without anymore bandwidth wastage.
- Jakes Model: Under certain assumptions we can adopt the Jakes model for the channel.
  - This allows us to have a second estimate independent of the data based (training sequence) estimate.
- The Kalman estimation algorithm uses these two independent estimates of the channel to produce a LMMSE estimate.
- Performance improvement: As a result of using the Jakes model in conjunction with the data based estimates there is a significant gain in the channel estimate.

Theoretical Background

Signal Multipath



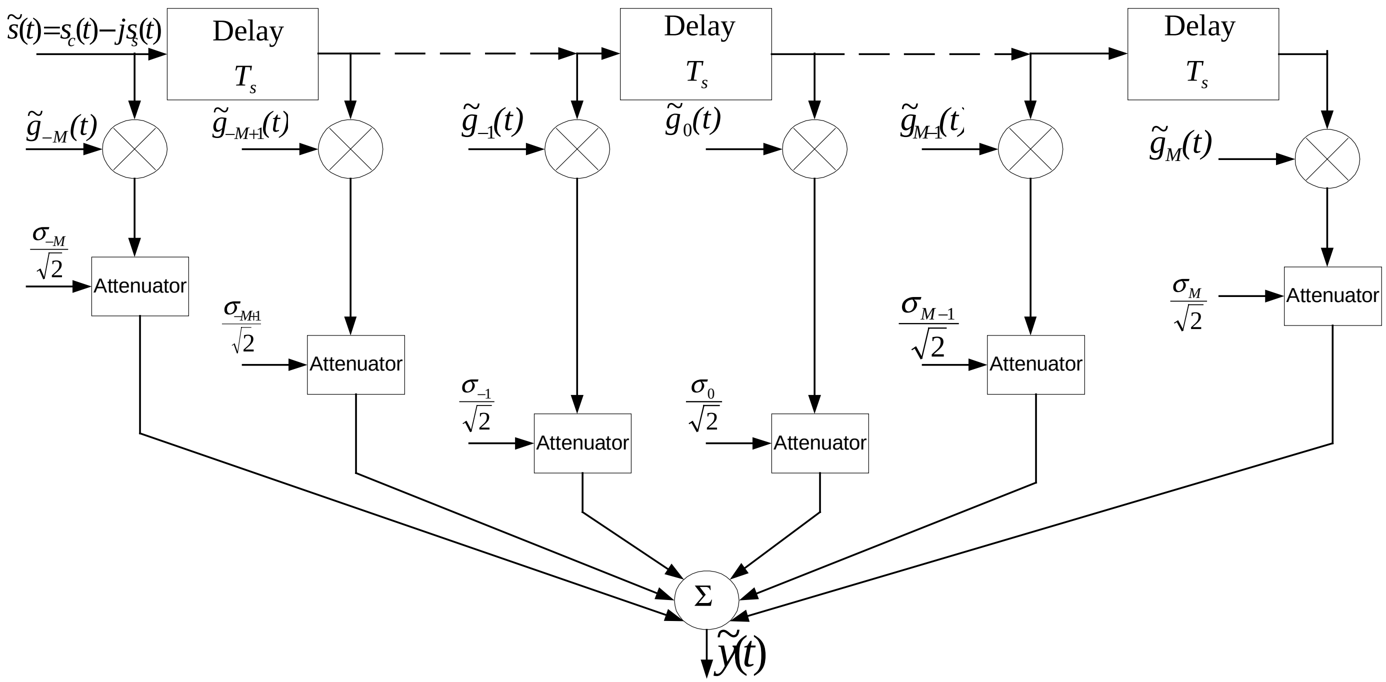
Multipath

- Signal multipath occurs when the transmitted signal arrives at the receiver via multiple propagation paths.
- Each path can have a separate phase, attenuation, delay and doppler shift associated with it.
- Due to signal multipath the received signal has certain undesirable properties like Signal Fading, Inter-Symbol-Interference, distortion etc.
- Two types of Multipath:
  - Discrete: When the signal arrives at the receiver from a limited number of paths.
  - Diffuse: The received signal is better modeled as being received from a very large number of scatterers.

Diffuse Multipath

- The Signal arrives via a continuum of multipaths. Thus the received signal is given by:
- $\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{\alpha}(\tau;t)\tilde{s}(t-\tau)d\tau$ where $\tilde{\alpha}(\tau,t)$ is the complex attenuation at delay τ and time t and $\tilde{s}(t)$ is the signal sent
- The Low Pass time variant impulse response is:
  - $\tilde{h}(\tau;t) = \alpha(\tau;t)e^{j2\pi f_C \tau}$ where f_C is the carrier frequency.
- If the signal is bandlimited then the channel can be represented as a tap-delayed line with time-varying coefficients and fixed tap spacing.

Tap Delayed Line Channel Model



Tap-Delay Line Model

- The received signal can be written as:
  - $\tilde{y}(t) = \sum_m \tilde{g}_m(t)\tilde{s}(t - \frac{m}{W})$ where:
  - $\tilde{g}_m(t) = \frac{1}{W}\tilde{h}(\frac{m}{W};t)$ is the sampled (in the τ domain) complex low-pass equivalent impulse response. *W*: is the Bandwidth of the Bandpass Signal

:

- WSSUS (Wide Sense Stationary Uncorrelated Scattering): Assuming WSSUS the delay profile and scattering function are as follows:
  - Multipath Intensity Profile: $R_C(\tau) = \frac{1}{2}E\left[\tilde{h}^*(\tau,t)\tilde{h}(\tau,t)\right]$ This defines the variation of average received power with delay. Delay spread is the range (in delay) for which the average power is non-zero.
  - Scattering function: $S(\tau;\nu) = F[R_C(\tau)]$This describes the power spectral density as a function of Doppler frequency (for fixed delay)

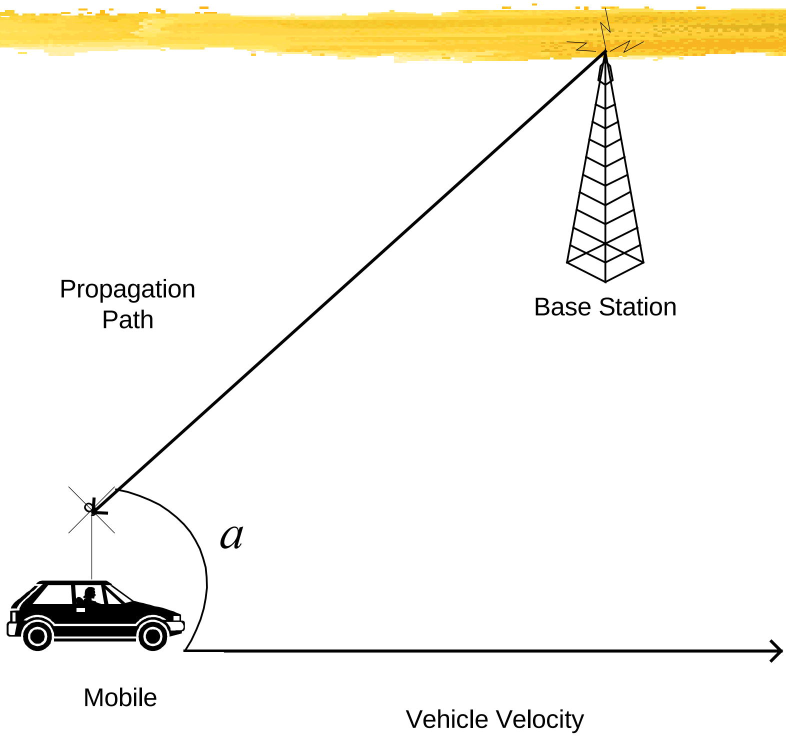
Tap gain functions

- Complex Gaussian process: Assuming infinite scatterers, as a consequence of the Central Limit Theorem, we can model the impulse response as a complex Gaussian process.
  - Rayleigh: If there is no one single dominant path, then the process is zero mean and the channel is Rayleigh fading.
  - Ricean: If there is asingle dominant path then the process is non-zero-mean and the channel is Ricean.
- The tap gain functions are then sampled complex gaussian processes.

Model Parameters

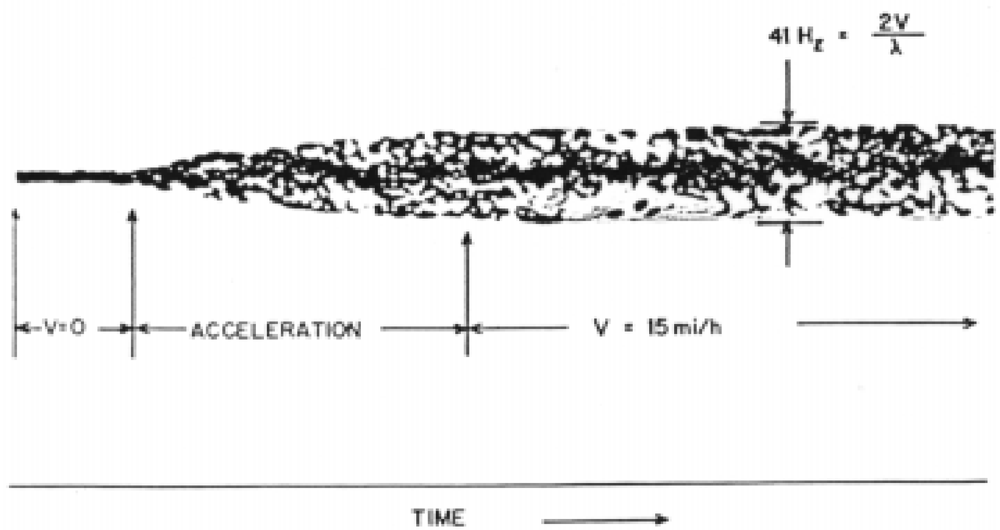
- The tap-delay model requires the following information.
  - Number of taps are $T_M W+1$. Where T_M is the delay spread and W is the information bandwidth.
  - Tap Spacing is: $1/W$.
  - Tap gain functions are discrete time complex Gaussian processes with variance given by the Multipth spread and PSD given by the scattering function.
- Tap gain functions as key: Once having specified the tap spacing and the number, it only remains to track the time varying tap gain functions in order to characterize the channel as modeled by the tap-delay line.
- Jakes model: The Jakes model (under certain assumptions) assigns the spectrum and autocorrelation to the tap-gain processes.

Jakes Model



- Assume plane waves are incident upon an omni-directional antenna from stationary scatterers.
- There will be a doppler shift induced in every wave.
  - Function of angle of arrival, carrier frequency and the receiver velocity.

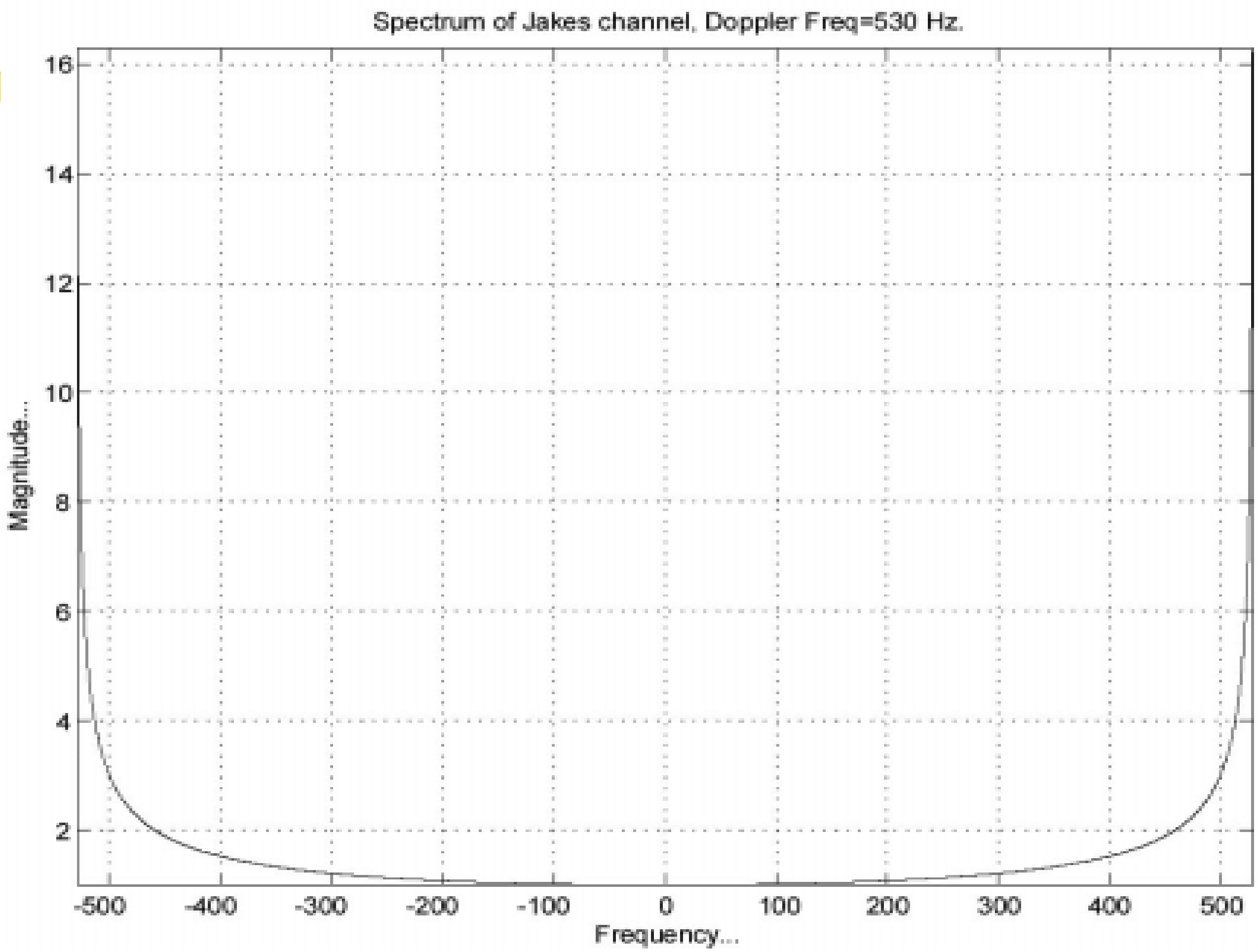
- Bounded Doppler Shift: The doppler shift is given by $f_d = \frac{v}{\lambda}\cos(\alpha)$ (where v is Vehicle velocity, λ , the wavelength of the carrier and α is the angle of arrival) ar



- Narrowband process: Since the Doppler spectrum is bounded, the electric field at the receiver is a narrowband process.
- Complex Gaussian Process: Assuming infinite scatterers and using the Central Limit theorem for narrowband processes, the received electric field is approximately a complex gaussian process.

Jakes Spectrum

- Uniform angle of arrival: Assume that the received power is uniformly distributed over the angle of arrival.
- Constant vehicle velocity: Finally the assumption is made that the vehicle is not accelerating.
- Power spectrum expression:
  - $S(f) = \left[1 - \left(\frac{f - f_C}{f_m}\right)^2\right]^{-1/2}$ where f_C is the carrier frequency and f_m is the doppler spread.
- Shaping Filter: The Jakes spectrum can be synthesized by a shaping filter with the following impulse response:
  - $h_j(t) = 2^{1/4}\Gamma(\frac{3}{4}) f_m (2\pi f_m t)^{-1/4} J_{1/4}(2\pi f_m t)$



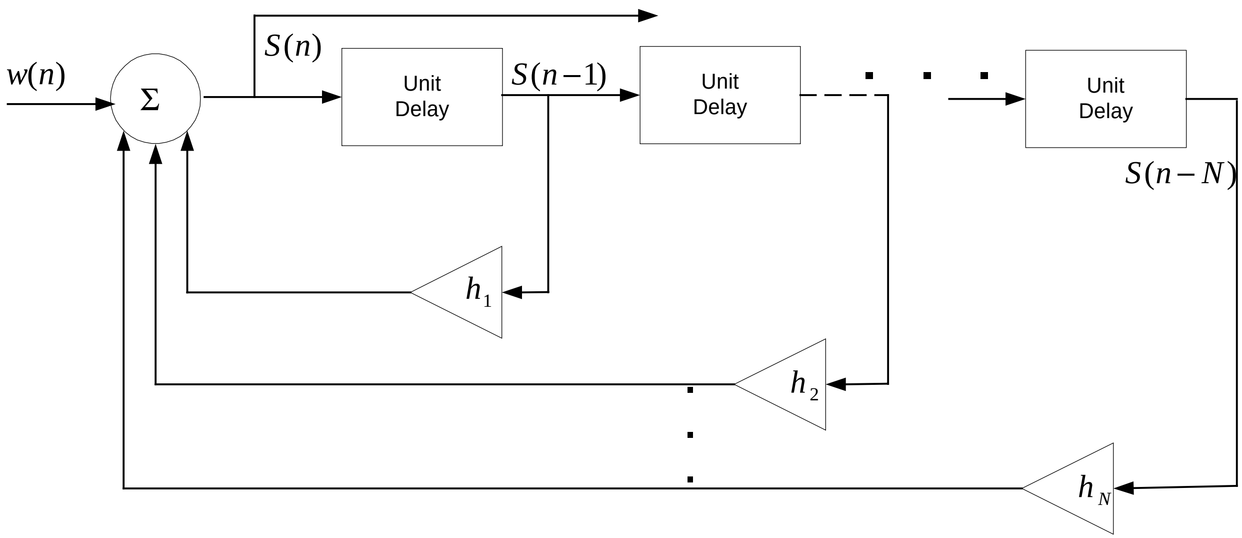
- Jakes Spectrum at a Doppler spread of 530 Hz.

The Channel Estimation Algorithm

Introduction

- Aim: To improve on the data-only estimate.
- Jakes model: We have adopted the Jakes model for the radio channel.
- Tap-gains as auto-regressive processes: The Jakes power spectrum is used to represent the tap-gains as AR processes.
- State-Space Representation: We have two independent estimates of the process from the data-based estimate and the Jakes model.
  - These are used to formulate a State-Space representation for the tap-gain processes.
  - An appropriate Kalman filter is derived from the state-space representation.
- Derivation: The algorithm is developed first for a Gauss-Markov Channel and then for the Jakes Multipath channel

AR representation of the Tap-gains



- General form: Any stationary random process can be represented as an infinite tap AR process.
- The current value is a weighted sum of previous values and the plant noise.

- Difference Equation form:
  - $S(n) = \sum_{i=1}^{N} \phi_i S(n-i) + w(n)$ Where *S(n):* is the complex gaussian process, ϕ_i are the parameters of the model. The model is driven by *w(n)*: A sequence of identically distributed zero-mean Complex Gaussian random variables.
- Solving for the AR parameters: This can be done in two ways
  - Yule-Walker equation solution: The autocorrelation coefficient of the process, is an N (number of taps in the AR model) order difference equation that can be solved.
  - PSD of the process: This method is more complex and involves finding the form of the shaping filter for the process PSD.

Data based estimator

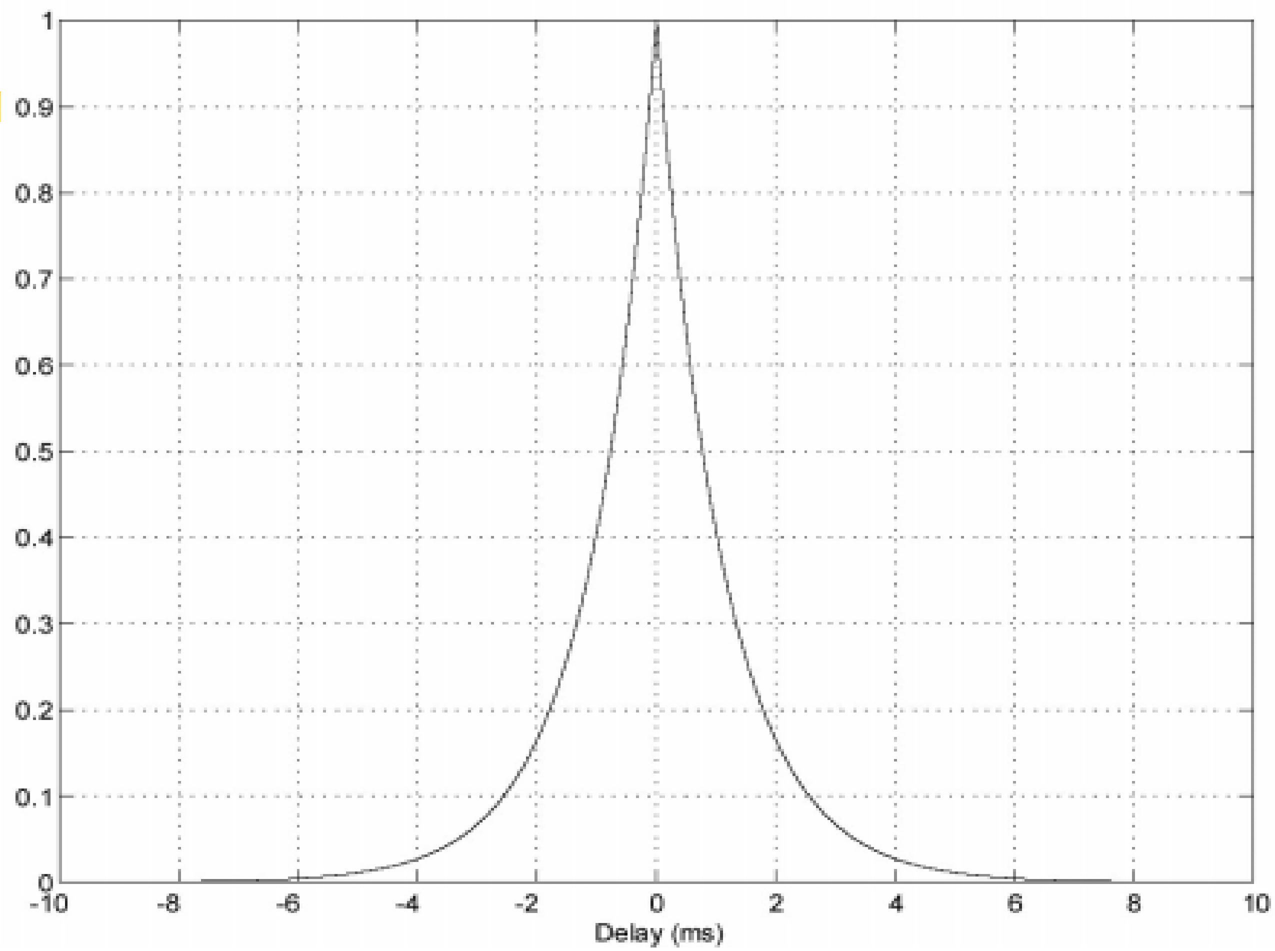
- Assumption: The channel remains constant over the span of the training sequence.
- Specifics:
  - Training sequence: Let the *'M'* length training sequence be:

    $\bar{x} = [x_0 x_1 \ldots\ldots\ldots\ldots\ldots x_{M-1}]^T$

  - Channel Impulse Response: Let the *'L'* length impulse response be $\bar{h} = [\tilde{h}_0 \tilde{h}_1 \tilde{h}_2 \ldots\ldots \tilde{h}_{L-1}]^T$
  - Received signal: For channel noise n_C of variance σ_C^2 the received signal in vector form is given by: $\bar{Y} = \bar{X}\,\bar{h} + \bar{n}_c$. Where $\bar{X}$ is an $(L+N-1 \times L)$ Toeplitz matrix containing delayed versions of the training sequence sent.

- Channel estimate: The data based estimate is given by correlating the received signal with the training sequence:
  - $\hat{\bar{h}} = (\bar{X}^T \bar{X})^{-1} (\bar{X}^T \bar{Y})$
- Estimation error: As expected the estimation error is a function of the channel noise. It is given by $\tilde{\bar{h}} = (\bar{X}^T \bar{X})^{-1} (\bar{X}^T \bar{n}_c)$
- Error Covariance: The performance of the data based estimator depends on the length of the training sequence and its autocorrelation:
  - $P_D = \sigma_c^2 (\bar{X}^T \bar{X})^{-1}$
  - For an ideal training sequence autocorrelation the error covariance is given by $P_D = \frac{\sigma_c^2}{M}$

Tracking a Gauss-Markov Channel



- A Gauss-Markov tap-gain process has an exponential autocorrelation.

Kalman Filter Derivation

- System Model: Since the auto-correlation is exponential, the tap gain is a simple first order AR process:
  - $S(n) = \phi_1 S(n-1) + w(n)$ Where

    $S(n)$: is the complex gaussian tap-gain process.

    ϕ_1 is the parameter of the AR model assumed.
- Observation Model: Since the data based estimator produces "noisy" estimates of the process. The following model emerges:
  - $X(n) = S(n) + v(n)$. Where

    $X(n)$ is the data based estimate of S(n)

    $v(n)$ is the error of the data based estimate and

    $\sigma_v^2 = \dfrac{\sigma_c^2}{M}$ is the error variance

Kalman filter equations

- Scalar Kalman filter: Given the state space representation, a standard scalar Kalman filter is used to track the process.
- Equations:
  - The initial conditions are:

$$\hat{S}(0)=E\{S(n)\}=0$$
$$P(1)\geq\left\{\sigma_w^2 \; and \; \sigma_v^2\right\}$$

  - The Kalman gain is given by:

$$k(n)=\frac{P(n)}{P(n)+\sigma_v^2(n)}$$

  - The current estimate of the process, after receiving the data estimate is given by:

$$\hat{S}_{curr}(n)=\hat{S}(n)+k(n)[X(n)-\hat{S}(n)]$$

  - The predicted estimate of the process is given by:

$$\hat{S}(n+1)=\phi_1\left\{\hat{S}_{curr}(n)\right\}$$

  - The current error covariance is given by:

$$P_{curr}(n)=[1-k(n)]P(n)$$

  - The prediction error covariance (MSE in this case) is given by:

$$P(n+1)=\phi_1^{\,2}\{P_{curr}(n)\}+\sigma_w^2$$

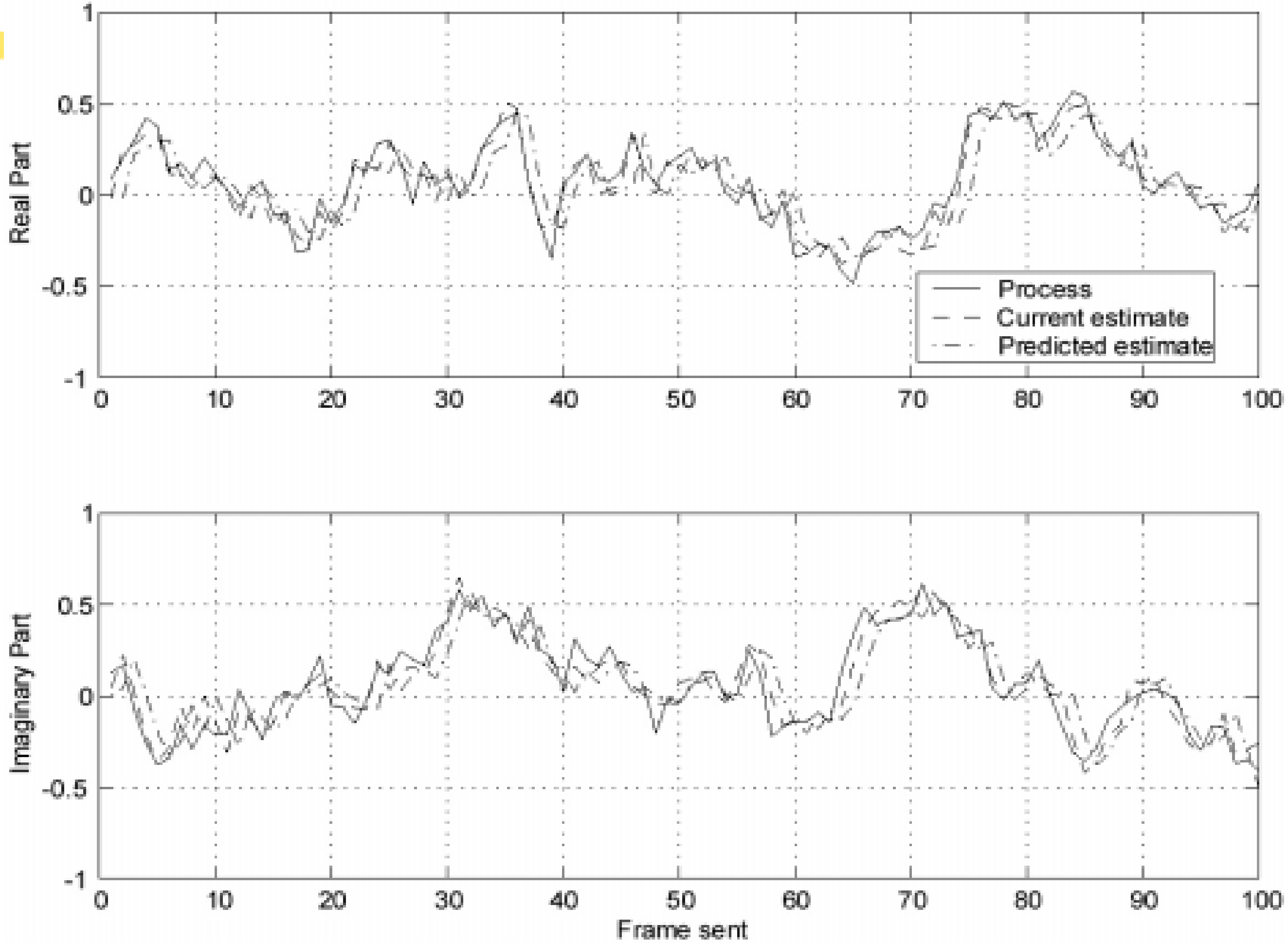
Simulation Parameters

- System equation: $S(n) = .9S(n-1) + w(n)$
- Observation Equation: $X(n) = S(n) + v(n)$
- The frame rate is : $R_F = 5 \times 10^4 \, Frames/\sec$. The simulation is run at the frame rate.
- $M = 8$: The length of the training sequence. The channel is assumed to be invariant for these M bits.
- The signal to noise ratio of the channel, for $E_B = 1$ is:

$$\frac{E_B}{N_O} = \frac{E_B}{2\sigma_c^2} = 6dB$$

- Thus $\sigma_c^2 = .1256$
- The data estimator variance $\sigma_V^2 = \frac{\sigma_c^2}{M} = 0.0157$
- The variance of the tap gain plant noise is $\sigma_w^2 = 2\sigma_V^2 = 0.0314$

Results:



Channel Estimation for a single ray Gauss-Markov channel

MSE for the estimator:

- Definition:
  - The current MSE is:

$$P_{curr} = E\left\{\left[S(n) - \hat{S}_{curr}(n)\right]\left[S(n) - \hat{S}_{curr}(n)\right]^H\right\}$$

  - The prediction MSE is:

$$P = E\left\{\left[S(n) - \hat{S}(n)\right]\left[S(n) - \hat{S}(n)\right]^H\right\}$$

- Steady State:
  - The steady state current error covariance is: $P_{curr_{SS}} = .0113$
  - The steady state prediction error covariance is: $P_{SS} = .0406$

- Simulation Results:
  - The current error covariance is: $P_{curr_{Sim}} = .0113$
  - The prediction error covariance is: $P_{Sim} = .0406$

- Performance Improvement: We can see that compared to the data only estimator, there is a significant improvement:
  - $\dfrac{\sigma_V^2 - P_{curr_{SS}}}{\sigma_V^2} = 28\%$

Tracking a single Jakes Tap-Gain Process

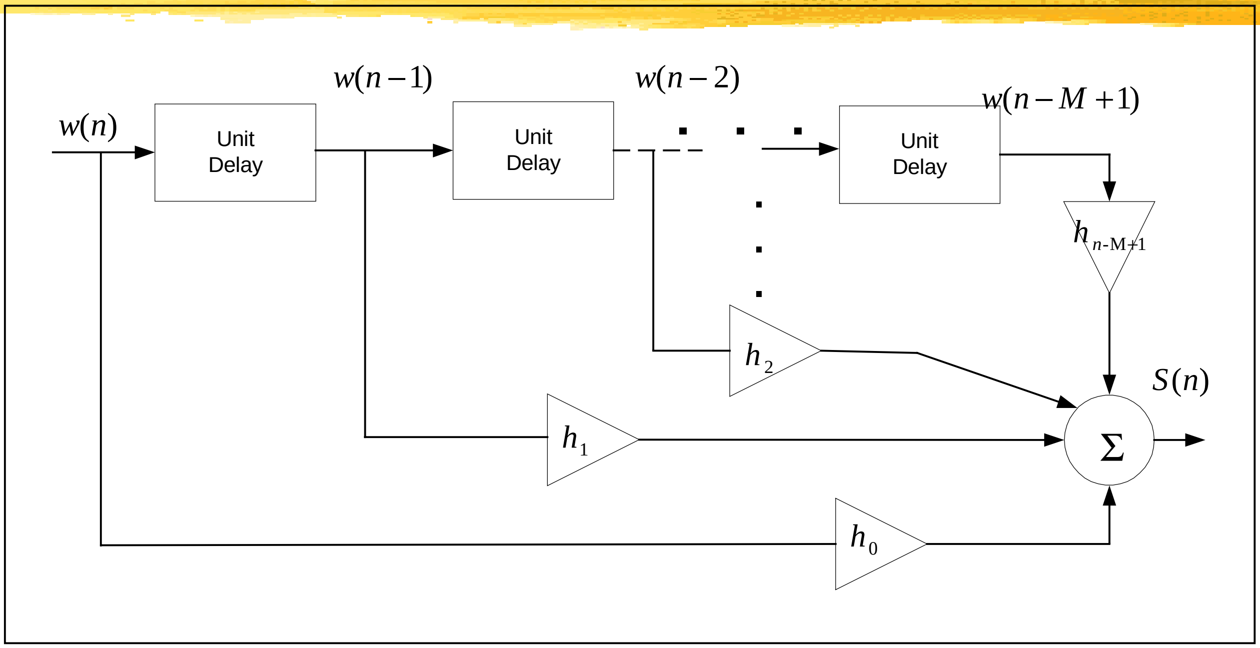
Single ray Jakes Channel

- Consider a single ray line of sight radio channel.
- More Complex Channel: The underlying channel model is no longer a single tap AR process.
- AR representation: The tap-gain process with the Jakes spectrum is a stationary process. We can represent it as an AR process.
  - Parameters: We derive the co-efficients for the process from the closed form expression of the Jakes channel-shaping filter.
- State-Space representation: Using the AR model and the data based estimator, a state-space representation is derived.
- Kalman tracking filter: Similar to the Gauss-Markov case, a Kalman filter to track the process is derived from the State-Space representation.

AR representation of the Jakes Process

- Jakes Shaping Filter: The closed form expression of the Jakes filter is given by:
- $h_j(t) = 2^{1/4}\Gamma(\frac{3}{4}) f_m (2\pi f_m t)^{-1/4} J_{1/4}(2\pi f_m t)$. Where Γ is the Gamma function and $J_{1/4}$ is the fractional Bessel function.
- FIR filter: This expression is sampled to produce the FIR Jakes channel shaping filter.
- Output: If the input to this filter is gaussian white noise, then the output is simply the convolution sum:
- $S(n) = \sum_{m=0}^{M-1} h_j(m) w(n-m)$. Where M is the liength of the FIR filter.

- Convolution as a MA sum: The convolution is nothing but a weighted moving average of the white noise inputs.

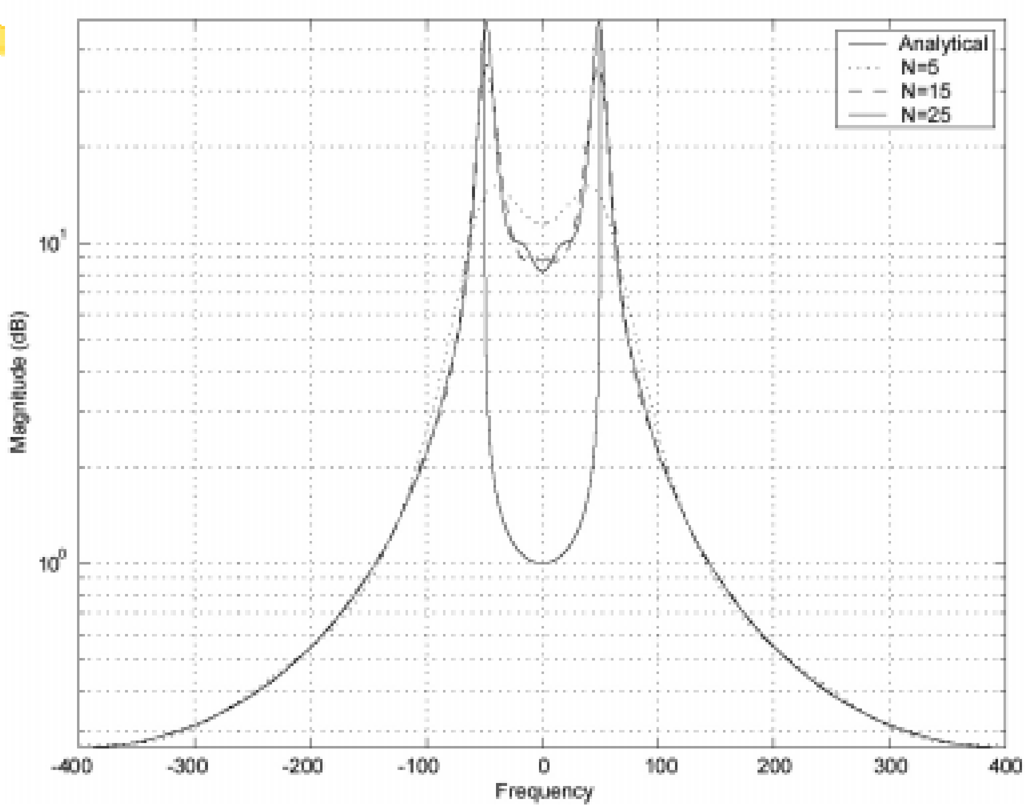


- Partial Fraction Inversion:
  - A finite order MA model can be represented as an infinite order AR series by the method of partial fractions as described by Box and Jenkins.
  - Order Truncation: Obviously for our purposes an infinite order AR model is impractical, so the infinite order AR model is truncated to order N.

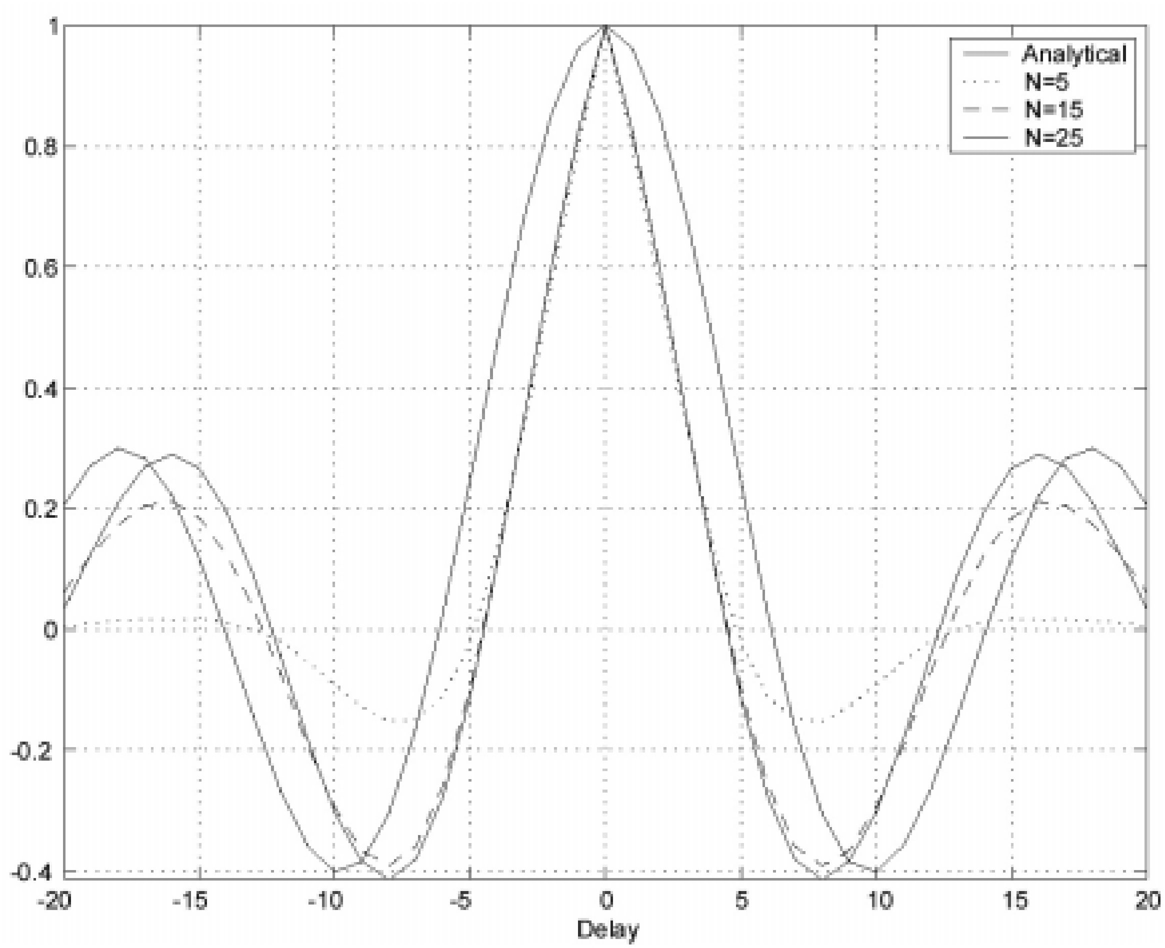
Model Validation

- Truncated AR model: The accuracy of the truncated models in representing the Jakes process is studied.
- Parameters:
  - $f_m = 50\,Hz$. This is the Doppler bandwidth of the channel.
  - $F_S = 16 \times f_m$. This is the Jakes-shaping-filter sampling rate.
  - $F_{S_{PSD}} = 5 \times 10^3\, samples\ /\,Hz$. This is the Jakes spectrum sampling frequency.
  - $N_{MA} = 64$. This is the FIR shaping filter length.
  - N is the length of the truncated AR model.
- PSD and autocorrelation:
  - PSD is given by the analytical expression: $S_{SS}(f) = \dfrac{\sigma_W^2}{\left|1 - \sum_{i=1}^{N} \phi_i e^{-j2\pi f i}\right|^2}$
  - Autocorrelation was estimated by actually creating the processes and finding their autocorrelations.

AR process length comparison



- Jakes spectrum for truncated AR processes.



- Jakes autocorrelation for truncated processes

Kalman filter Derivation

- State Space representation: As in the previous case, we are going to represent the system using the State-Space form and derive the appropriate Kalman filter.
- System model: The AR form of the Jakes process is
  - $S(n) = \sum_{i=1}^{N} \phi_i S(n-i) + w(n)$. Where ϕ_i are the AR coefficients calculated and N is the order of the model used.
  - Its two equivalent forms are:

$$\begin{bmatrix} S(n) \\ S(n\text{-}1) \\ . \\ . \\ S(n\text{-}N+1) \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & . & . & \phi_N \\ 1 & . & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \\ 0 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} S(n\text{-}1) \\ S(n\text{-}2) \\ . \\ . \\ S(n\text{-}N) \end{bmatrix} + \begin{bmatrix} w(n) \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$ in matrix form

and $\overline{S}(n) = A\overline{S}(n-1) + \overline{W}(n)$ in vector form.

- The system matrix is defined as:

- $$A = \begin{bmatrix} \phi_1 & \phi_2 & . & . & \phi_N \\ 1 & . & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \\ 0 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- The Plant noise covariance matrix is

- $$Q = E\left[\bar{W}(n)\left(\bar{W}(n)\right)^H\right] = \begin{bmatrix} \sigma_w^2 \quad 0 \quad 0 \ldots 0 \\ 0 \; 0 \ldots\ldots\ldots \; 0 \\ 0 \; 0 \; 0 \ldots\ldots \; 0 \\ 0 \; 0 \ldots\ldots\ldots \; 0 \end{bmatrix}$$

- Observation Equation:
  - $X(n) = S(n) + v(n)$ Where *v(n)* is the error of the data based estimate with a variance of $\sigma_v^2 = \frac{\sigma_c^2}{M}$
  - Expressed in matrix and vector form:
  - $$\begin{bmatrix} X(n) \\ X(n\text{-}1) \\ . \\ . \\ X(n\text{-}N+1) \end{bmatrix} = \begin{bmatrix} S(n) \\ S(n\text{-}1) \\ . \\ . \\ S(n\text{-}N+1) \end{bmatrix} + \begin{bmatrix} v(n) \\ v(n\text{-}1) \\ . \\ . \\ v(n\text{-}N+1) \end{bmatrix}$$ or $\bar{X}(n) = H \times \bar{S}(n) + \bar{v}(n)$

    where *H* is the identity matrix.

- Observation noise covariance matrix:
  - $$R = E\left[\bar{v}(n)\left(\bar{v}(n)\right)^H\right] = \sigma_v^2 \times \begin{bmatrix} 1 & 0 & 0.......0 \\ 0 & 1.......... & .0 \\ 0 & 0 & 1.........0 \\ 0 & 0 & ....1 \end{bmatrix}$$

- Kalman filter Equations: Given the above State Space formulation, we can use a vector Kalman filter to track the tap gain process.
  - The initial conditions are:

$\hat{S}(0) = E\{S(n)\}$ =zero matrix of length $(N \times L)$

$$P(1) \geq \left\{\sigma_w^2 [I] and \sigma_v^2 [I]\right\}$$

  - The Kalman gain is given by:

$$K(n) = P(n)H^T[HP(n)H^T + R]^{-1}$$

  - The current estimate of the process, given the data estimate is given by:

$$\hat{S}_{curr}(n) = \hat{S}(n) + K(n)[X(n) - H\hat{S}(n)]$$

  - The predicted estimate of the process, is given by:

$$\hat{S}(n+1) = A\left\{\hat{S}_{curr}(n)\right\}$$

  - The current error covariance is given by:

$$P_{curr}(n) = [I - K(n)H]P(n)$$

  - The predicted error covariance is given by:

$$P(n+1) = A\{P_{curr}(n)\}A^T + Q$$

Simulation Parameters

- System equation:

$$\bar{S}(n) = A\bar{S}(n-1) + \overline{W}(n)$$

- $N=5$. This is the number of taps in the AR model.

- $$\phi = [.9086, -0.0590, -0.0548, -0.0486, -0.0409]$$

- $$A = \begin{bmatrix} .9086 & -0.0590 & -0.0548 & -0.0486 & -0.0409 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- Observation Equation:

$$\overline{X}(n) = H \times \bar{S}(n) + \bar{v}(n)$$

- The Doppler bandwidth is $f_m = 500Hz$

- The frame rate is : $R_F = 5 \times 10^4 \, Frames/\sec$. The simulation is run at the frame rate.

- $M = 8$: The length of the training sequence.
- The signal to noise ratio of the channel, for $E_B = 1$ is:

$$\frac{E_B}{N_O} = \frac{E_B}{2\sigma_c^2} = 6dB$$

Thus $\sigma_c^2 = .1256$

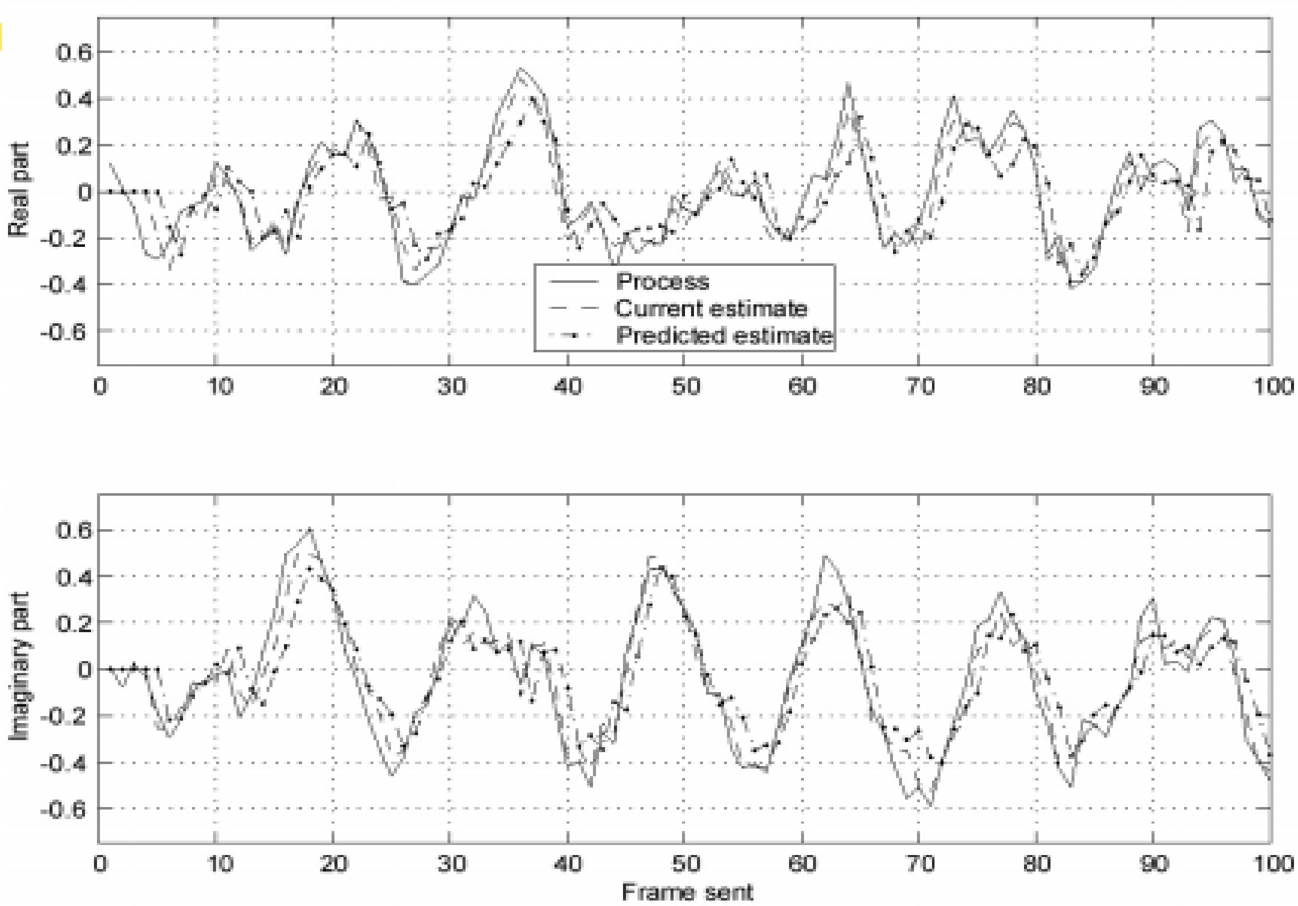
- The variance of the tap gain plant noise is $\sigma_w^2 = 2\sigma_V^2 = 0.0314$

$$Q = 0.0314 \times \begin{bmatrix} 1 & 0 & 0......0 \\ 0 & .... & 0 \\ 0 & 0........ & .0 \\ 0 & 0 .... & 0 \end{bmatrix}$$

- The data estimator variance $\sigma_V^2 = \frac{\sigma_c^2}{M} = 0.0157$

$$R = .0157. \times \begin{bmatrix} 1 & 0 & 0...0 \\ 0 & 1........... & 0 \\ 0 & 0\ 1........ & 0 \\ 0 & 0 .... & 1 \end{bmatrix}$$

Results



- Channel Estimation of a single ray Jakes channel

Error covariance:

- The error covariance is defined as:

$$P = E\left\{\left[S(n+1)-\overset{\wedge}{\mathrm{S}}(\mathrm{n}+1)\right]\left[S(n+1)-\overset{\wedge}{\mathrm{S}}(\mathrm{n}+1)\right]^H\right\}$$

- We can then interpret the diagonal elements as follows:

$$P = \begin{bmatrix} MSE_{(n+1/n)} \; ** \ldots\ldots * \\ * \quad MSE_{(n/n)} \; ** \ldots\ldots * \\ * \quad\quad * \quad MSE_{(n-1/n)} \; ** \ldots\ldots * \\ . \\ .** \ldots\ldots\ldots\ldots\ldots\ldots \quad MSE_{S(n-N+2/n)} \end{bmatrix}$$

Where $MSE_{(n+1/n)}$ is the MSE of the prediction.

$MSE_{(n/n)}$ is the MSE of the current states estimate

- The steady state and simulated error covariance matrices are:
  - $diag(P_{SS}) = \begin{bmatrix} 1.0921 \\ 0.1117 \\ 0.0565 \\ 0.0388 \\ 0.0296 \end{bmatrix}$ $\quad$ $diag(P_{Sim}) = \begin{bmatrix} 1.0777 \\ 0.1125 \\ 0.1081 \\ 0.1118 \\ 0.1144 \end{bmatrix}$
- **Performance Improvement**: There is a significant performance gain compared to the data only estimator
  - $\dfrac{\sigma_V^2 - diag(P_{Sim})(2,1)}{\sigma_V^2} = 29\%$

Multipath Channel Estimation

Multipath Jakes Channel

- Consider a multipath radio channel.
- Assume the Jakes model on each path.
- AR representation: For the Multipath case, a modification of the single ray AR system model is presented.
- State-Space representation: Using the AR model and the data based estimator, a state-space representation is derived.
- Kalman tracking filter:Once again a vector Kalman filter is used to track the tap-gain functions.

System model

- Assumptions: The tap gain processes are independent but have the same Jakes spectrum.
- AR Representation:

$$S_1(n) = \sum_{i=1}^{N} \phi_i S_1(n-i) + w_1(n)$$

$$S_2(n) = \sum_{i=1}^{N} \phi_i S_2(n-i) + w_2(n)$$

.

.

$$S_L(n) = \sum_{i=1}^{N} \phi_i S_L(n-i) + w_L(n)$$

where

- $S_l(n)$ is the l^{th} process to be tracked
- ϕ_i are the AR model parameters. These are the same for each process.
- $w_l(n)$ is the plant noise driving the tap gain function. The relative variance is determined by the power delay profile of the channel.
- L is the number of processes being tracked

- The matrix form of the system equation for 'L' processes is:

$$\begin{bmatrix} S_1(n)\ldots\ldots\ldots\ldots S_L(n) \\ S_1(n-1)\ldots\ldots\ldots S_L(n-1) \\ . \qquad\qquad . \\ . \\ S_1(n-N+1)\ldots S_L(n-N+1) \end{bmatrix} = \begin{bmatrix} \phi_1\phi_2\ldots\ldots\phi_N \\ 1\ \ 0\ \ 0\ldots\ldots 0 \\ 0\ \ 1\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots 0 \\ 0\ \ 0\ldots\ldots 1\ \ 0 \end{bmatrix} \begin{bmatrix} S_1(n-1)\ldots\ldots S_L(n-1) \\ S_1(n-2)\ldots\ldots S_L(n-2) \\ . \qquad\qquad . \\ . \\ S_1(n-N)\ldots S_L(n-N) \end{bmatrix} + \begin{bmatrix} w_1(n)\ldots\ldots w_L(n) \\ 0\ldots\ldots\ldots\ldots\ldots 0 \\ . \qquad\qquad . \\ . \\ 0\ldots\ldots\ldots\ldots\ldots 0 \end{bmatrix}$$

- In vector form: $\bar{S}(n) = A\bar{S}(n-1) + \bar{W}(n)$, where

  - $A = \begin{bmatrix} \phi_1\phi_2\ldots\ldots\phi_N \\ 1\ \ 0\ \ 0\ldots\ldots 0 \\ 0\ \ 1\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots 0 \\ 0\ \ 0\ldots\ldots 1\ \ 0 \end{bmatrix}$ is the system matrix.

  - $Q = E\left[\bar{W}(n)\left(\bar{W}(n)\right)^H\right] = \begin{bmatrix} \sum_{l=1}^{L} \sigma_w^2(l)\ldots\ldots 0 \\ 0\ \ldots\ldots\ldots\ldots\ldots 0 \\ 0\ \ldots\ldots\ldots\ldots\ldots 0 \\ . \\ 0\ \ldots\ldots\ldots\ldots\ldots 0 \end{bmatrix}$ is the plant noise covariance matrix.

Observation Model

- Assuming that the data based estimates are path-wise independent, we have the following model:

$$X_1(n) = S_1(n) + v_1(n)$$
$$X_2(n) = S_2(n) + v_2(n)$$
$$.$$
$$.$$
$$X_L(n) = S_L(n) + v_L(n)$$

Where,

  - $S_l(n)$: The l'th process at time n.
  - $X_l(n)$: The l'th data based estimate of $S(n)$
  - $v_l(n)$: Error of the l'th data based estimate.
  - $\sigma_v^2 = \frac{\sigma_c^2}{M}$ is assumed to be the same for all paths

- The Observation Equation can be written in matrix and vector form as follows:
  - $$\begin{bmatrix} X_1(n) & . & . & X_L(n) \\ X_1(n-1) & . & . & X_L(n-1) \\ . & . & . & . \\ . & . & . & . \\ X_1(n-N+1) & . & . & X_L(n-N+1) \end{bmatrix} = \begin{bmatrix} S_1(n) & . & . & S_L(n) \\ S_1(n-1) & . & . & S_L(n-1) \\ . & . & . & . \\ . & . & . & . \\ S_1(n-N+1) & . & . & S_L(n-N+1) \end{bmatrix} + \begin{bmatrix} v_1(n) & . & . & v_L(n) \\ v_1(n-1) & . & . & v_L(n-1) \\ . & . & . & . \\ . & . & . & . \\ v_1(n-N+1) & . & . & v_L(n-N+1) \end{bmatrix}$$
  - $\bar{X}(n) = H \times \bar{S}(n) + \bar{v}(n)$ where H is an identity matrix.
- The observation noise covariance matrix is given by:
  - $R = E[v(n)(v(n))^H] = (L\sigma_v^2) \times [I]$ where $[I]$ is an $(N \times N)$ identity matrix.

Simulation parameters

- N=5. This is the number of taps in the AR model.
- L=3: This is the number of tap-gain processes being tracked.
- $$\phi = [.9086, -0.0590, -0.0548, -0.0486, -0.0409]$$
- The System Equation

$$A = \begin{bmatrix} .9086 & -0.0590 & -0.0548 & -0.0486 & -0.0409 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- The Doppler bandwidth is $\mathrm{f_m} = 500Hz$
- The frame rate is : $R_F = 5 \times 10^4 \, Frames/\sec$. The simulation is run at the frame rate.
- $M = 8$: The length of the training sequence. The channel is assumed to be invariant for these M bits.

- The signal to noise ratio of the channel, for $E_B = 1$ is:

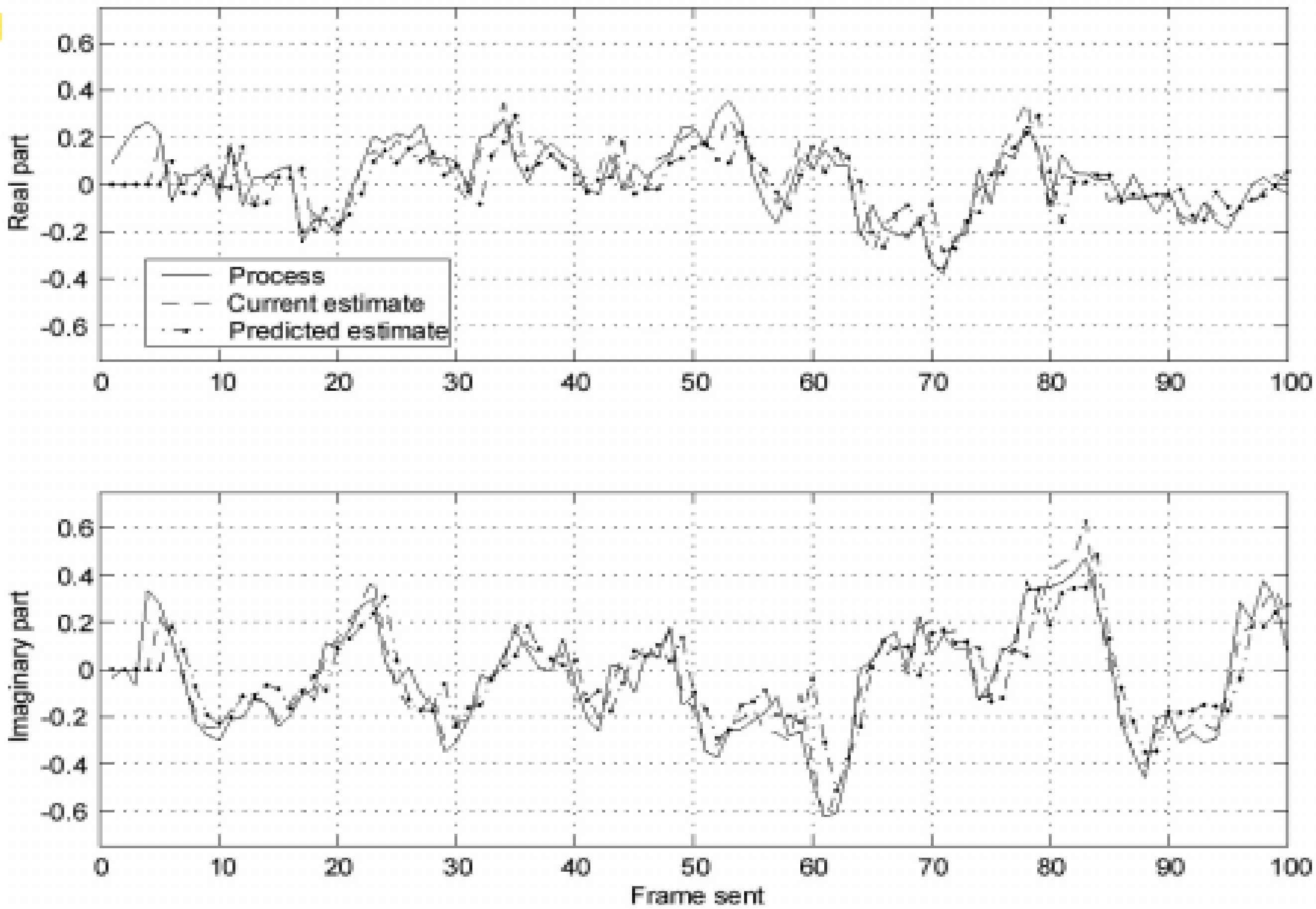
$$\frac{E_B}{N_O} = \frac{E_B}{2\sigma_c^2} = 6dB$$

- Thus $\sigma_c^2 = .1256$
- The power delay profile $\sigma_w^2(l) = 0.0314 \times [1, 0.9, 0.81]$
- The covariance of the plant noise is:

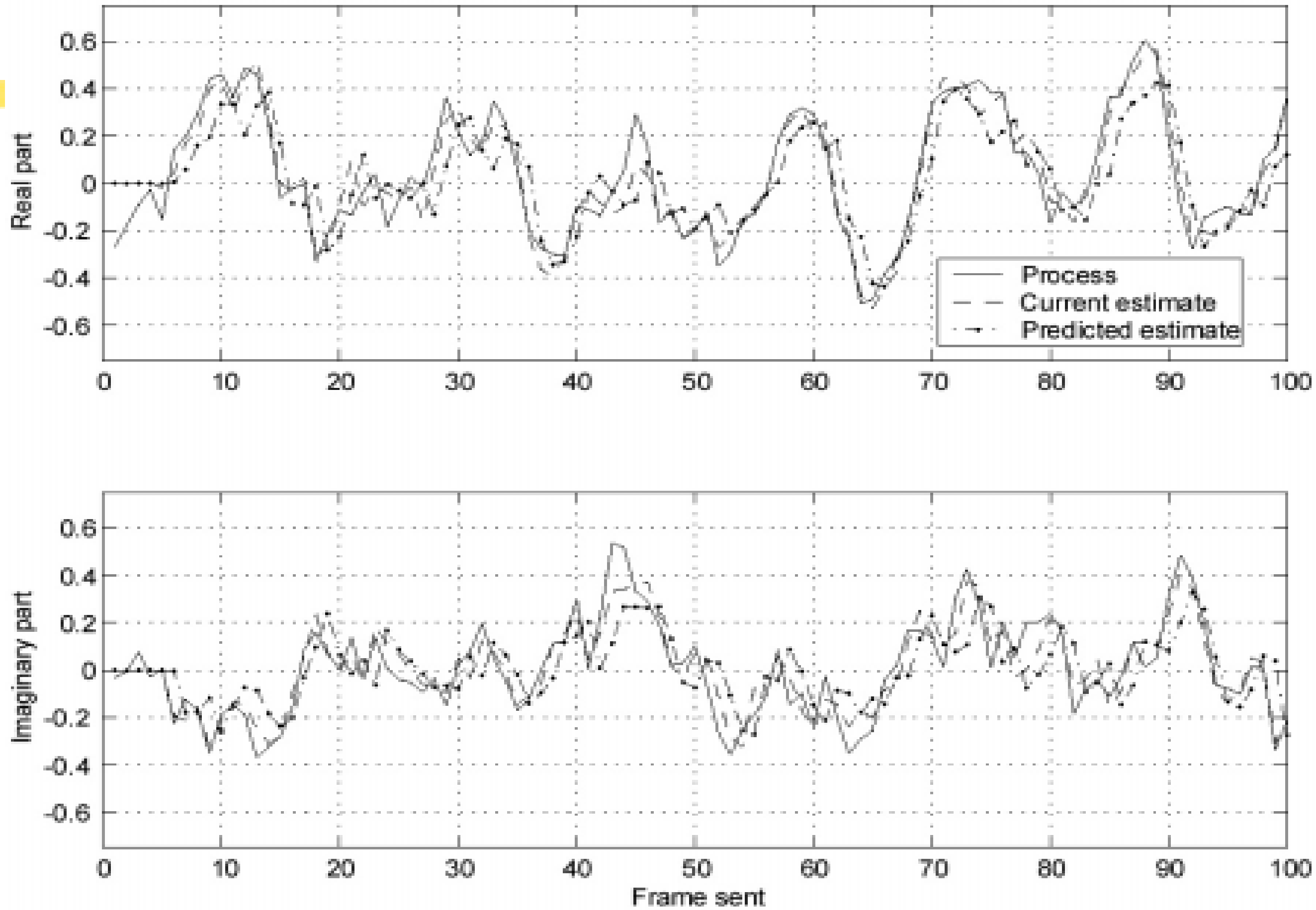
$$Q = \begin{bmatrix} 0.0851 & . & 0 \\ . & . & . \\ 0 & . & 0 \end{bmatrix}$$

- The data estimator variance $\sigma_v^2 = \frac{\sigma_c^2}{M} = 0.0157$. Here the assumption is made that the training sequence has an ideal autocorrelation.
  - $R = 3 \times 0.0157. \times [\mathrm{I}]$

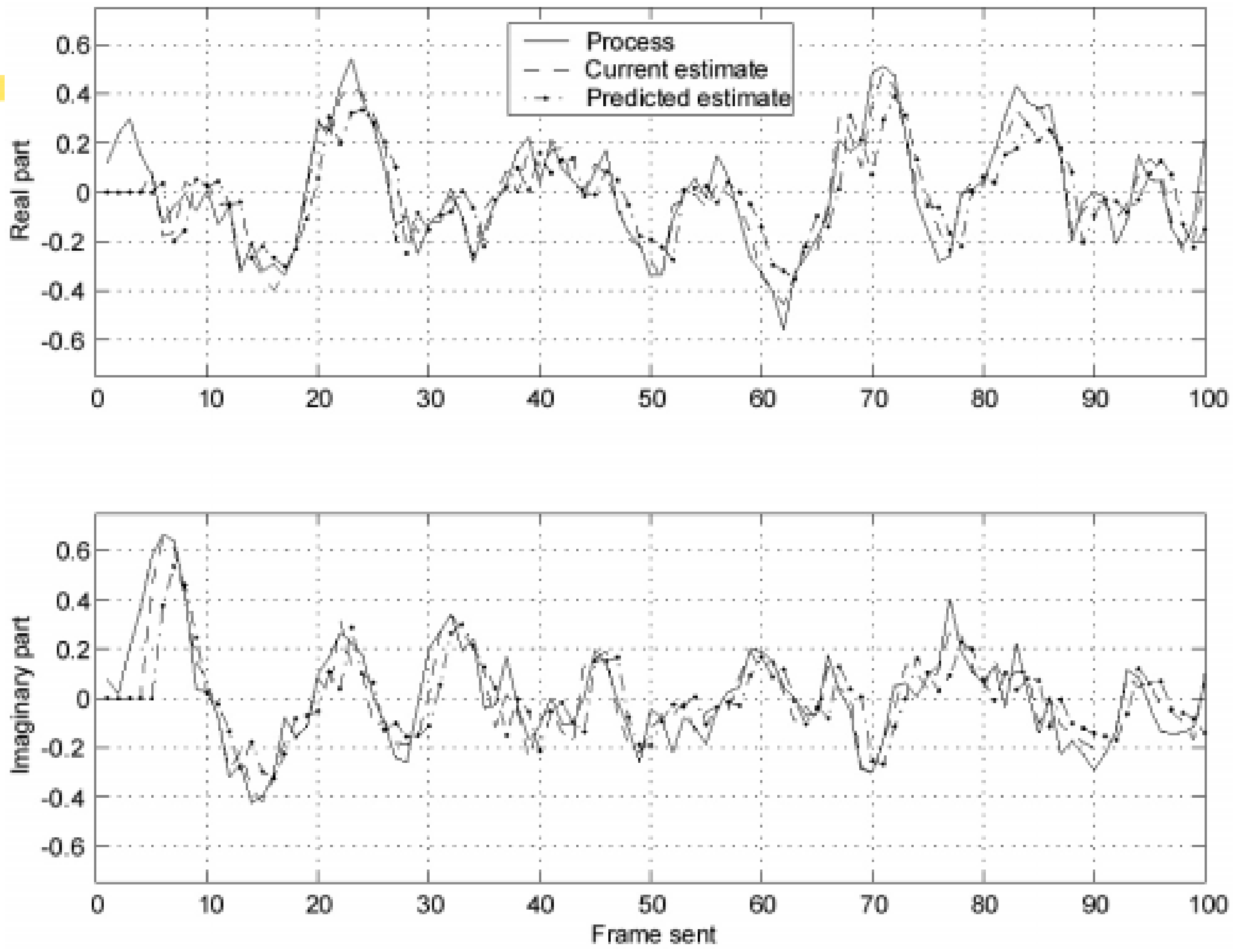
Results



- Channel estimation for the first process.



- Channel estimation for the second process.



- Channel estimation for the third process.

Error Covariance

- The Error covariance is defined as:
  - $P = E\left\{\left[S(n+1) - \hat{S}(n+1)\right]\left[S(n+1) - \hat{S}(n+1)\right]^H\right\}$
  - We can then interpret the diagonal elements as follows:

$$P = \begin{bmatrix} \sum_l MSE_{(n+1/n)}(l) & * & * & \ldots\ldots. & * \\ * & \sum_l MSE_{(n/n)}(l) & * & * & \ldots\ldots. * \\ * & * & \sum_l MSE_{(n-1/n)}(l) & * & * \ldots\ldots. * \\ . & & & & \\ . * * \ldots\ldots\ldots\ldots\ldots\ldots & & & \sum_l MSE_{(n-N+2/n)}(l) & \end{bmatrix} \text{where}$$

- $\sum_l MSE_{(n+1/n)}(l)$: the sum of the MSE of the predicted state estimate on all processes.

- $\sum_l MSE_{(n/n)}(l)$: the MSE of the current states estimate on all processes.

- The simulated error covariance diagonal is given by:

$diag(P_{Sim}(l))$	Process # (l) 1	2	3	$diag(P_{Sim})$	$diag(P_{SS})$
	0.0384	0.0351	0.0321	0.1056	0.1111
	0.0112	0.0109	0.0106	0.0327	0.0321
	0.0098	0.0097	0.0097	0.0292	0.0169
	0.0105	0.0105	0.0105	0.0315	0.0122
	0.0112	0.0111	0.0112	0.0335	0.0096
$diag(P_{Curr_{Sim}}(l))$				$diag(P_{Curr_{Sim}})$	$diag(P_{Curr_{SS}})$
	0.0112	0.0109	0.0106	0.0327	0.0321
	0.0098	0.0097	0.0097	0.0292	0.0169
	0.0105	0.0105	0.0105	0.0315	0.0122
	0.0112	0.0111	0.0112	0.0335	0.0096
	0.0117	0.0117	0.0117	0.0351	0.008

- **Performance improvement** on each path:
  - The data based estimate has a MSE of $\sigma_V^2 = 0.0157$ on each path.
  - Path 1 improvement: $\dfrac{\sigma_V^2 - diag(P_{curr_{sim}})(1,1)}{\sigma_V^2} = \dfrac{.0157 - .0112}{.0157} \times 100 = 28.66\%$
  - Path 2 improvement: $\dfrac{.0157 - .0109}{.0157} \times 100 = 30.57\%$
  - Path 3 improvement: $\dfrac{.0157 - .0106}{.0157} \times 100 = 32.48\%$
- Almost a 30% improvement on each path

Conclusions

- Developed a Kalman filter based channel estimation algorithm for the Multipath radio channel.
- Significant gain in performance over a training sequence based estimator.
- This improvement is obtained without wasting any more bandwidth.
- Also allows us to predict the channel state without having to wait for data.

Future Work

- Use Of Multiple Sampling Rates:
  - Instead of waiting for the data to arrive at the end of every frame we can run the Kalman filter at a higher rate than the frame rate.
  - In the absence of a data based estimate perform the time-update portion of the algorithm and do a measurement update when data is received.
  - Allows estimates to be available as required.
- Different process models on each path:
  - In case the process model varies with path, we can still use the Kalman filter but with some modifications to the system matrix.
- Correlated paths:
  - For correlated paths the Kalman filter needs to be modified.