Image Theory for Electromagnetic Sources in Chiral Medium Above the Soft and Hard Boundary

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Abstract—The classical image theory valid for electromagnetic (EM) sources in an isotropic medium above a planar perfect electric conductor (PEC) or perfect magnetic conductor (PMC) surface was more recently extended to involve the planar soft-and-hard surface (SHS) boundary that can be realized with tuned corrugations. The image principle is now generalized to EM sources in isotropic chiral medium above a SHS boundary. The problem is solved by two consecutive decompositions of the sources reducing the problem to four classical ones involving electric and magnetic sources above PEC and PMC boundaries; each involving an isotropic nonchiral medium and possessing a known image solution. One of the decompositions is based on the fact that the two eigenwaves of a chiral medium do not couple at a soft-and-hard surface; and, the other one, on the eigendepolarizations of the reflection dyadic.

Index Terms—Image theory, soft-and-hard surfaces, chiral media.

I. INTRODUCTION

Electromagnetic (EM) problems involving a planar perfect electric (PEC) or perfect magnetic conductor (PMC) can be classically handled through the image principle in which the surface is replaced by image sources that basically are mirror images of the sources with a positive or negative sign [1]. A similar principle for the planar soft-and-hard surface (SHS) boundary was derived in [2]. In this case the image of a dipole was seen to consist of a rotated dipole at the mirror image point plus an extended source of transmission-line type. The SHS boundary concept was introduced in [6], [7] as one of the basic boundaries in electromagnetics. It can be realized by a tuned corrugated surface that has been of interest in electromagnetics since its original introduction in slow-wave structures [3] and later in horn antennas radiating axially symmetric patterns [4], [5]. In the present paper the image principle is extended to the more general case when the medium above the SHS boundary is isotropic and chiral. The motivation for this study is the basic nature of the problem that has not been previously treated to the knowledge of this author. Chiral medium is known to affect the polarization of the propagating EM field even if the medium itself is isotropic.

The soft-and-hard surface is characterized by the following symmetric boundary conditions for the electric and magnetic fields

\[ \mathbf{v} \cdot \mathbf{E} = 0 \quad \mathbf{v} \cdot \mathbf{H} = 0 \]  

where \( \mathbf{v} \) is a unit vector defining a fixed direction on the surface. It is of interest to note that SHS belongs to the more general class of ideal boundaries defined so that the complex Poynting vector has no normal component [8].

The class of isotropic chiral media is characterized by the constitutive relations

\[ \mathbf{D} = \varepsilon \mathbf{E} + j \kappa_\tau \sqrt{\mu} \mathbf{H} \]  
\[ \mathbf{B} = \mu \mathbf{H} - j \kappa_\tau \sqrt{\mu} \mathbf{E} \]

where \( \kappa_\tau \) is the dimensionless relative chirality parameter. In the lossless case \( \kappa_\tau \) is real and satisfies the condition \( |\kappa_\tau| < 1 \) [10].

II. SELF-DUAL DECOMPOSITION

Fields in the chiral medium can be decomposed in two parts known in different sources as wavefields, self-dual fields, or Bohren-decomposed fields [9]

\[ \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \quad \mathbf{H} = \mathbf{H}_+ + \mathbf{H}_- \]

The self-dual components \( \mathbf{E}_+ , \mathbf{H}_+ , \mathbf{E}_- , \mathbf{H}_- \) are two EM fields noncoupled in a homogeneous chiral medium and they can be expressed as

\[ \mathbf{E}_\pm = \frac{1}{2} (\mathbf{E} \mp j \eta \mathbf{H}) \]  
\[ \mathbf{H}_\pm = \frac{1}{2} (\mathbf{H} \pm j \frac{\eta}{\eta} \mathbf{E}) = \pm j \frac{\eta}{\eta} \mathbf{E}_\pm . \]

They are generated by the respective decomposed EM sources \( \mathbf{J}_\pm, \mathbf{M}_\pm \)

\[ \mathbf{J}_\pm = \frac{1}{2} \left( \mathbf{J} \mp j \frac{\eta}{\eta} \mathbf{M} \right) \]  
\[ \mathbf{M}_\pm = \frac{1}{2} (\mathbf{M} \pm j \eta \mathbf{J}) = \pm j \eta \mathbf{J}_\pm . \]

Here \( \eta = \sqrt{\mu/\varepsilon} \) is an impedance quantity. In fact, the self-dual fields satisfy the noncoupled Maxwell equations of the form [9]

\[ \nabla \times \mathbf{E}_\pm = -j \omega \mu_\pm \mathbf{H}_\pm - \mathbf{M}_\pm \]  
\[ \nabla \times \mathbf{H}_\pm = j \omega \varepsilon_\pm \mathbf{E}_\pm + \mathbf{J}_\pm \]

when the parameters of the two equivalent isotropic nonchiral media are defined by

\[ \varepsilon_\pm = \varepsilon (1 \pm \kappa_\tau) \quad \mu_\pm = \mu (1 \pm \kappa_\tau) \]

corresponding to two different wavenumbers

\[ k_\pm = k (1 \pm \kappa_\tau) \quad k = \omega \sqrt{\mu \varepsilon} \]

but the same impedance \( \eta_\pm = \eta \).
It is also easily seen that the two self-dual fields do not couple at the SHS boundary. In fact, inserting (4) in (1) and noting the connection (6) leads to the conditions

\[
v \cdot E_+ = 0, \quad v \cdot H_+ = 0, \quad v \cdot E_- = 0, \quad v \cdot H_- = 0. \tag{13}
\]

Thus, for example, the source \( J_+ , M_+ \) creates only the self-dual field component \( E_+ , H_+ \) in the chiral medium terminated by a SHS boundary. In this case, the chiral medium can be replaced by the equivalent nonchiral medium with the parameters \( \epsilon_+ , \mu_+ \). Decomposition of the original source \( J, M \) in two parts (7), (8) is, thus, seen to give the possibility to split the original problem involving a chiral medium in two problems involving nonchiral media.

### III. IMAGE THEORY

#### A. TE/TM Decomposition

The self-dual field decomposition makes it now possible to apply the previously derived image theory [2] to the chiral medium without having to do the tedious plane-wave analysis similar to that in [11]. The SHS image theory was based on a TE/TM decomposition of fields with respect to the direction \( \mathbf{v} \) on the SHS boundary. In fact, it was shown in [13] that plane waves reflected from a SHS plane have TE and TM polarizations as eigenpolarizations and for these two polarizations the SHS plane can be replaced by equivalent PMC and PEC planes, respectively. Since this is valid for any plane waves, it is valid for any field that can be expressed as a sum or integral of plane waves, i.e., any field outside the sources.

If we can decompose the electric and magnetic sources into parts radiating TE and TM fields only in homogeneous space, the original problem can be reduced to finding images of sources in nonchiral media in the presence of planar PEC or PMC boundaries, which is the classical problem. How to decompose a given source to its TE and TM components with respect to any given direction in space has earlier been considered in [12].

#### B. Decomposition of Electric Dipole

As an example of this method, let us assume that the original source is an electric dipole of moment \( IL \) at height \( z = h \) and direction defined by the unit vector \( \mathbf{v} \),

\[
J(\mathbf{r}) = vIL \delta(\mathbf{r} - \mathbf{u}_z h) = vIL \delta(x) \delta(y) \delta(z - h). \tag{14}
\]

This is first decomposed in two pairs of self-dual sources as

\[
J_+(\mathbf{r}) = vIL_2 \delta(x) \delta(y) \delta(z - h) \tag{15}
\]

\[
M_+(\mathbf{r}) = v\frac{mI}{2} \delta(x) \delta(y) \delta(z - h)
\]

\[
J_-(\mathbf{r}) = vIL_2 \delta(x) \delta(y) \delta(z - h) \tag{16}
\]

\[
M_-(\mathbf{r}) = v\frac{mI}{2} \delta(x) \delta(y) \delta(z - h).
\]

Each pair of sources lies in its own equivalent nonchiral medium corresponding to the two wavenumbers \( k_{\pm} = k(1 \pm \kappa_+). \) We can consider images of all four sources separately, although, only together they make sense in the chiral medium.

Let us for simplicity now assume that \( \mathbf{v} = \mathbf{u}_z \) equals the unit vector along the \( x \) coordinate axis. If the dipole is parallel to the \( x \) axis, the sources \( J_\pm \) produce pure TM fields and \( M_\pm \) produce TE fields. In this simple case there is no need for the TE/TM decomposition. To have a more general example, let us assume that this is not the case and, in particular, that the dipole is orthogonal to the \( x \) axis: \( \mathbf{v} \cdot \mathbf{u}_z = 0. \) Invoking now the decomposition expressions from [2], we can write

\[
J_\pm(\mathbf{r}) = J_{\pm}^{TM} + J_{\pm}^{TE} \tag{17}
\]

with

\[
J_{\pm}^{TM} = -\mathbf{u}_z \frac{IL}{2} G_{\pm}(x) \mathbf{v} \cdot \nabla \delta(z - h) \delta(y) \tag{18}
\]

\[
J_{\pm}^{TE} = \mathbf{v} \frac{IL}{2} \delta(x) \delta(y) \delta(z - h) + \mathbf{u}_z \frac{IL}{2} G_{\pm}(x) \mathbf{v} \cdot \nabla \delta(z - h). \tag{19}
\]

Here, \( \mathbf{v} \cdot \nabla \) operates on both \( \delta(y) \) and \( \delta(z - h) \) in general. \( G_{\pm}(x) \) denote derivatives of two one-dimensional Green functions,

\[
G_{\pm}(x) = \frac{1}{2jk_\pm} e^{-jk_\pm |\mathbf{r}|} \tag{20}
\]

The decomposed sources are bifilar line-current waves of transmission-line type. From duality, we can decompose the two magnetic dipoles as

\[
M_\pm(\mathbf{r}) = M_{\pm}^{TM} + M_{\pm}^{TE} \tag{21}
\]

\[
M_{\pm}^{TE} = \pm \mathbf{u}_z \frac{mIL}{2} G_{\pm}(x) \mathbf{v} \cdot \nabla \delta(y) \delta(z - h) \tag{22}
\]

\[
M_{\pm}^{TM} = \pm \mathbf{v} \frac{mIL}{2} \delta(x) \delta(y) \delta(z - h) \tag{23}
\]

involving bifilar magnetic line currents. As a check at this point, we can see that summing all four components of the electric current gives us the original current \( \mathbf{J} \) while the magnetic current components sum up to zero.

#### C. Image Expressions

The images of the decomposed electric and magnetic currents in the PEC and PMC planes can be expressed as [14]

\[
J_{\pm}^{TM}(\mathbf{r}) = -\tilde{C} \cdot J_{\pm}^{TM}(\tilde{C} \cdot \mathbf{r}) \tag{24}
\]

\[
J_{\pm}^{TE}(\mathbf{r}) = -\tilde{C} \cdot J_{\pm}^{TE}(\tilde{C} \cdot \mathbf{r}) \tag{25}
\]

\[
M_{\pm}^{TM}(\mathbf{r}) = \tilde{C} \cdot M_{\pm}^{TM}(\tilde{C} \cdot \mathbf{r}) \tag{26}
\]

\[
M_{\pm}^{TE}(\mathbf{r}) = -\tilde{C} \cdot M_{\pm}^{TE}(\tilde{C} \cdot \mathbf{r}) \tag{27}
\]

Here \( \tilde{C} = \mathbf{I} - 2\mathbf{u}_z \mathbf{u}_z \) denotes the mirror image dyadic that reverts the \( z \) component of any vector. It is worth noting that the original dipole has actually been decomposed into eight sources before the image theory can be applied. As was seen in [2], in the nonchiral case, decomposition into two sources was sufficient.
Applying these expressions to those of the decomposed dipole, we obtain

\[ J_{\pm}^T(r) = -\mathbf{\mathcal{E}} \cdot \mathbf{v} \frac{IL}{2} \delta(x) \delta(y) \delta(z + h) \]
\[ - u_x \frac{IL}{2} G^\prime_{\pm}(x) \mathbf{v} \cdot \mathbf{\mathcal{E}} \cdot \nabla \delta(y) \delta(z + h) \]  
(28)

\[ J_{\pm}^M(r) = - u_z \frac{IL}{2} G^\prime_{\pm}(x) \mathbf{v} \cdot \mathbf{\mathcal{E}} \cdot \nabla \delta(y) \delta(z + h) \]  
(29)

\[ M_{\pm}^T(r) = \mp u_x \frac{jnIL}{2} G^\prime_{\pm}(x) \mathbf{v} \cdot \mathbf{\mathcal{E}} \cdot \nabla \delta(y) \delta(z + h) \]  
(30)

\[ M_{\pm}^M(r) = \mp \mathbf{\mathcal{E}} \cdot \frac{jnIL}{2} G^\prime_{\pm}(x) \mathbf{v} \cdot \mathbf{\mathcal{E}} \cdot \nabla \delta(y) \delta(z + h). \]  
(31)

Expressing the unit vector \( \mathbf{v} \) in component form as

\[ \mathbf{v} = v_y \mathbf{u}_y + v_z \mathbf{u}_z \]  
(32)

the differential operation gives

\[ \mathbf{v} \cdot \mathbf{\mathcal{E}} \cdot \nabla \delta(y) \delta(z + h) \]
\[ = v_y \delta^\prime(y) \delta(z + h) - v_z \delta(y) \delta^\prime(z + h). \]  
(33)

The TE and TM image sources can be combined in each of the two equivalent media as

\[ J_{\pm}(r) = J_{\pm}^T(r) + J_{\pm}^M(r) \]
\[ = \mathbf{\mathcal{E}} \cdot \left[ J_{\pm}^T(\mathbf{\mathcal{E}} \cdot \mathbf{r}) - J_{\pm}^T(\mathbf{\mathcal{E}} \cdot \mathbf{r}) \right] \]
\[ - u_x 2ILG^\prime_{\pm}(x) \mathbf{v} \cdot \mathbf{\mathcal{E}} \cdot \nabla \delta(y) \delta(z + h) \]  
(34)

\[ M_{\pm}(r) = M_{\pm}^T(r) + M_{\pm}^M(r) \]
\[ = \mathbf{\mathcal{E}} \cdot \left[ M_{\pm}^T(\mathbf{\mathcal{E}} \cdot \mathbf{r}) - M_{\pm}^T(\mathbf{\mathcal{E}} \cdot \mathbf{r}) \right] \]
\[ \mp u_x 2ILG^\prime_{\pm}(x) \mathbf{v} \cdot \mathbf{\mathcal{E}} \cdot \nabla \delta(y) \delta(z + h) \]  
(35)

which reduces the number of images to four. It is easy to check that these image sources satisfy the relation (8), which means that the images of self-dual sources are also self-dual sources.

The self-dual sources give rise to the same EM fields when they exist in their respective equivalent nonchiral media with the parameters \( (\varepsilon_+ + j\mu_+), (\varepsilon_-, j\mu_-) \) or in the original chiral medium with the parameters \( (\varepsilon, j\mu, k) \). This is the same for both self-dual sources; we can combine all image sources and assume that they all reside in the original chiral medium. Thus, we obtain the following expressions for the total image components:

\[ J_{\pm}(r) = J_{\pm}(r) + J_{\pm}(r) \]
\[ = \mathbf{\mathcal{E}} \cdot \left[ J_{\pm}^T(\mathbf{\mathcal{E}} \cdot \mathbf{r}) - J_{\pm}^T(\mathbf{\mathcal{E}} \cdot \mathbf{r}) \right] \]
\[ + J_{\pm}^M(\mathbf{\mathcal{E}} \cdot \mathbf{r}) - J_{\pm}^M(\mathbf{\mathcal{E}} \cdot \mathbf{r}) \]  
(36)

\[ M_{\pm}(r) = M_{\pm}(r) + M_{\pm}(r) \]
\[ = \mathbf{\mathcal{E}} \cdot \left[ M_{\pm}^T(\mathbf{\mathcal{E}} \cdot \mathbf{r}) - M_{\pm}^T(\mathbf{\mathcal{E}} \cdot \mathbf{r}) \right] \]
\[ + M_{\pm}^M(\mathbf{\mathcal{E}} \cdot \mathbf{r}) - M_{\pm}^M(\mathbf{\mathcal{E}} \cdot \mathbf{r}) \]  
(37)

Inserting the expressions above we have

\[ J_{\pm}(r) = -\mathbf{\mathcal{E}} \cdot \mathbf{v} IL \delta(r + u_r h) \]
\[ - u_x 2ILG^\prime_{\pm}(x) \mathbf{v} \cdot \mathbf{\mathcal{E}} \cdot \nabla \delta(y) \delta(z + h) \]
\[ \text{(38)} \]
\[ M_{\pm}(r) = 0, \]
\[ \text{(39)} \]

Here \( G^\prime(x) \) denotes the function

\[ G^\prime(x) = \frac{1}{2} \left( G^+(x) + G^-(x) \right) \]
\[ = -\frac{\text{sgn}(x)}{4} \left( e^{-jkx} |\mathbf{d}| + e^{-j(k-\gamma) |\mathbf{d}|} \right) \]
\[ = -\frac{\text{sgn}(x)}{2} e^{-jkx} \cos(\kappa_x k |\mathbf{d}|). \]
\[ \text{(40)} \]

As seen from above, the magnetic component of the image of an electric source vanishes. This is to be expected, because for an electric source the self-dual magnetic source components \( M_+ \) and \( M_- \) are the same except for the sign, thus, their images are bound to cancel.

From duality, we can write similar image expressions for the magnetic dipole

\[ M(r) = \mathbf{v} ML \delta(r - u_r h) \]
\[ M_{\pm}(r) = -\mathbf{\mathcal{E}} \cdot \mathbf{v} ML \delta(r + u_r h) \]
\[ - u_x 2MLG^\prime_{\pm}(x) \mathbf{v} \cdot \mathbf{\mathcal{E}} \cdot \nabla \delta(y) \delta(z + h) \]
\[ \text{(42)} \]
\[ J_{\pm}(r) = 0. \]
\[ \text{(43)} \]

As a check of the result we can easily see that for a nonchiral medium \( (\kappa_r = 0) \) the image expressions (38), (39) reduce to that of the previous theory [2], because the cosine term in (40) reduces to unity.

IV. CONCLUSION

Image theory has been constructed for the case when an EM source is in chiral medium above a planar soft-and-hard boundary. Because the conditions corresponding to such a boundary are known to be self dual, they do not couple the self-dual ‘plus’ and ‘minus’ fields in a chiral medium. As an example of the general expressions given by the theory, more detailed image formulas for electric and magnetic dipoles were presented. Developing a corresponding image method for a source in chiral medium bounded by a PEC or PMC boundary would become much more complicated because in these cases the boundary presents coupling between the self-dual fields.

REFERENCES


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