Theory of Noisy Fourpoles*

H. ROTHE†, SENIOR MEMBER, IRE, AND W. DAHLKE†

Summary—The well-known theory of fourpoles only comprises passive fourpoles and active fourpoles with internal sources of sinusoidal currents or voltages of defined frequencies. This theory is now completed for fourpoles with internal noise sources. Simple equivalent circuits are derived for such networks. They consist of the original but noise-free fourpole cascaded with a preceding noise fourpole in which all noise-sources are concentrated. The latter contains the equivalent noise conductance G_n, the equivalent noise resistance R_n and the complex correlation admittance Y_{cor}. With these quantities the noise behavior of any desired fourpole can be described sufficiently. In particular it is possible to calculate the noise figure F and its dependence on the matching conditions to the signal source of a single fourpole or a group of cascaded fourpoles. The methods of experimental determination of the elements of the noise fourpoles are discussed. The same theory is also useful for mixer-circuits as well as for traveling-wave tubes and transistors, as application results are given for grid controlled electron tubes.

Fourpole Equations of Noisy Fourpoles

In Fig. 1(a) is shown the principal scheme of a fourpole with internal noise sources. The electrical behavior of this fourpole will be described by two linear equations between the input voltage and current U_1 and I_1 and the output voltage and current U_2 and I_2. The special form of these equations depends on the network itself, if the Π-admittance matrix or the T-resistance matrix is more convenient as equivalent circuit. But if there are inner noise sources inside the fourpole, these well-known fourpole equations are no more sufficient. They must rather be completed by two noise currents i_1 and i_2 respectively, by two noise voltages u_1 and u_2 to the form

$$I_1 = Y_{11}U_1 + Y_{12}U_2 + i_1$$
$$U_1 = Z_{11}I_1 + Z_{12}I_2 + u_1$$
$$\text{or} \tag{1}$$
$$I_2 = Y_{21}U_1 + Y_{22}U_2 + i_2$$
$$U_2 = Z_{21}I_1 + Z_{22}I_2 + u_2.$$

Here i_1 and i_2 (u_1 and u_2) represent the short circuit

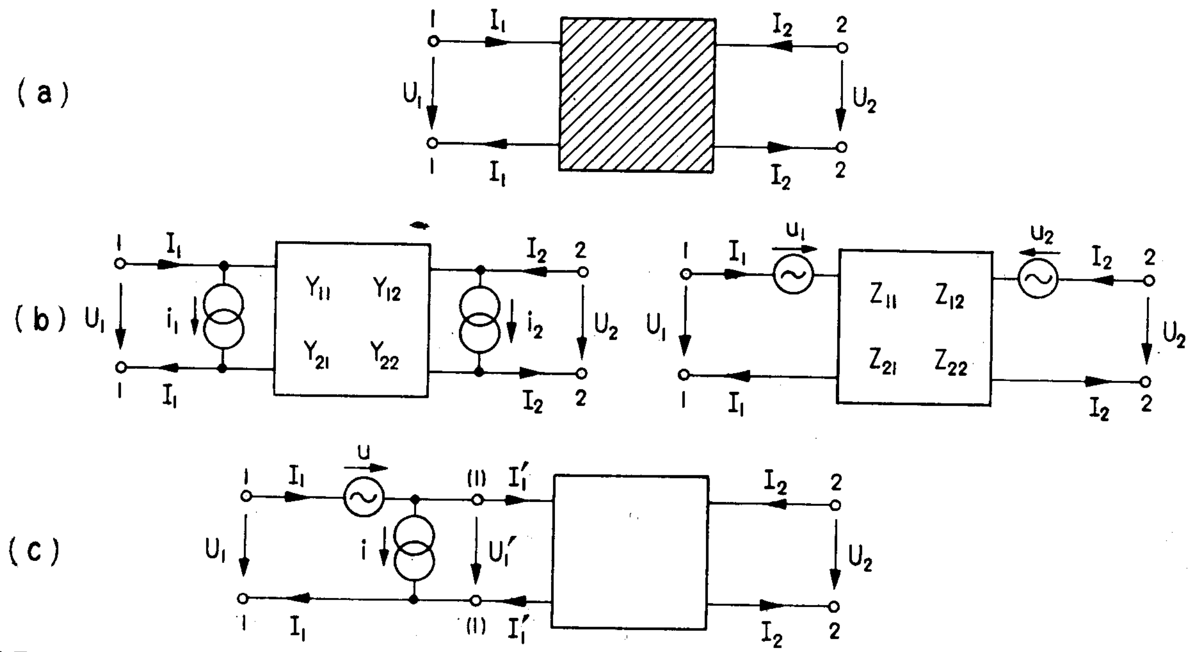

Fig. 1—(a) Fourpole with internal noise sources; (b) equivalent circuit with the outside noise current sources i_1 and i_2 respectively u_1 and u_2; (c) equivalent circuit with noise voltage source u and noise current source i at the input.

noise current (open circuit noise voltage) at the input respectively at the output for $U_1 = U_2 = 0$ ($I_1 = I_2 = 0$) caused only by the internal noise sources. Between both noise sources normally a correlation has to be assumed.

The system of Fig. 1 can be represented by the equivalent circuits of Fig. 1(b). In these circuits the noisy four-pole of Fig. 1(a) is replaced by a noise-free but otherwise unchanged fourpole together with the noise current sources i_1 and i_2 (noise voltage sources u_1 and u_2) with an inner infinite (zero) impedance.

In order to characterize the noise qualities of a fourpole it is more convenient to use only noise sources preceding the noise-free fourpole. This is possible by using

* Original manuscript received by the IRE August 15, 1955; revised manuscript received November, 1955. Previously published in the "Archiv der elektrischen Übertragung," vol. 9, pp. 117–121, 1955. In extract discussed by the IRE Fall Meeting, Syracuse, October, 1954, and in the Symposium on Fluctuations in Microwave Tubes, New York, November, 1954.
† Telefunken G.m.b.H., Ulm/Donau, Germany.

the chain matrix

$$I_1 = AU_2 + BI_2 + i,$$
$$U_1 = CU_2 + DI_2 + u. \tag{2}$$

All internal noise sources then will be represented at the input side by a noise current source i and a noise voltage source u, as shown in the equivalent circuit of Fig. 1(c). It consists of the noise-free fourpole between the points *(1)(1)* and *2 2* and a preceding noise fourpole between the points *1 1* and *(1)(1)*.

$$u = -i_2/Y_{21}$$
$$i = i_1 + uY_{11} = i_1 - i_2(Y_{11}/Y_{21})$$
$$\text{or } u = u_1 + iZ_{11} = u_1 - u_2(Z_{11}/Z_{21})$$
$$i = -u_2/Z_{21}. \tag{4}$$

As equivalent of a noise current source i_2 (noise voltage source u_2) at the output, therefore, a voltage source u (current source i) and an additional current source $-i_2Y_{11}/Y_{21}$ (voltage source $-u_2 Z_{11}/Z_{21}$) are necessary at the input.

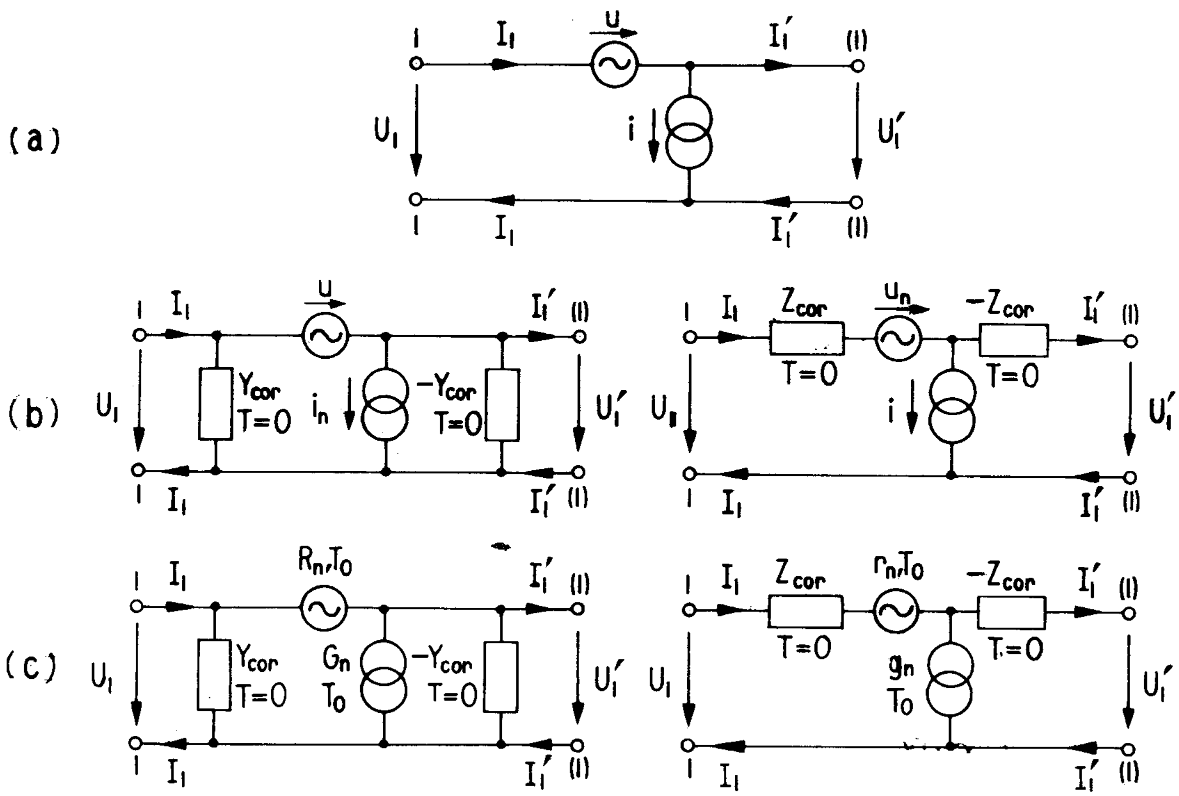

Fig. 2—(a) Noise fourpole with correlated noise sources u and i; (b) noise fourpole with correlation admittance Y_{cor} (correlation impedance Z_{cor}) and uncorrelated noise voltage source u, noise current source i_n (u_n and i); (c) noise fourpole with correlation admittance Y_{cor} (correlation impedance Z_{cor}) and uncorrelated noise sources R_n and G_n (r_n and g_n).

With the current i_1' and the voltage U_1' at the input terminals *(1)(1)* of the noise-free fourpole Eq. (2) changes to

$$I_1 = I_1' + i,$$
$$U_1 = U_1' + u. \tag{2a}$$

By introducing the noise sources i and u in Eq. (1) we get

$$I_1 = Y_{11}(U_1 - u) + Y_{12}U_2 + i$$
$$I_2 = Y_{21}(U_1 - u) + Y_{22}U_2$$
$$\text{or } U_1 = Z_{11}(I_1 - i) + Z_{12}I_2 + u$$
$$U_2 = Z_{21}(I_1 - i) + Z_{22}I_2. \tag{3}$$

A comparison with Eq. (1) gives the transforming formulas for both of the new noise sources

Noise Fourpole and Characteristic Noise Terms

Normally a correlation exists between the two noise sources u and i of the above defined noise fourpole shown in Fig. 2(a). But the noise current i (noise voltage u) can be divided into one part i_n (u_n) not correlated to $u(i)$ and a second part fully correlated to u (i). This second part must be proportional to u (i). As factor of proportionality having the dimension of an admittance (impedance) we introduce the complex correlation admittance $Y_{cor} = G_{cor} + jB_{cor}$ (correlation impedance $Z_{cor} = R_{cor} + jX_{cor}$). We therefore may write

$$i = i_n + uY_{cor}, \quad \text{or} \quad u = u_n + iZ_{cor}. \tag{5}$$

These new terms Y_{cor} and Z_{cor} corresponding to the well-known correlation coefficient

$$\gamma = \frac{\overline{iu^*}}{\sqrt{\overline{|i|^2}\,\overline{|u|^2}}}$$

are defined by (4, 5, 11)

$$Y_{cor} = \gamma\sqrt{\frac{\overline{|i|^2}}{\overline{|u|^2}}} = \frac{\overline{iu^*}}{\overline{|u|^2}}$$

$$\text{or} \quad Z_{cor} = \gamma\sqrt{\frac{\overline{|u|^2}}{\overline{|i|^2}}} = \frac{\overline{i^*u}}{\overline{|i|^2}}. \tag{6}$$

The correlation between i_1 and $u(u_1$ and $i)$ in (4) which is not identical with the correlation between i and u, can be expressed by another correlation coefficient

$$\alpha = \frac{\overline{i_1u^*}}{\sqrt{\overline{|i_1|^2}\,\overline{|u|^2}}} \quad \text{resp.} \quad \frac{\overline{u_1i^*}}{\sqrt{\overline{|u_1|^2}\,\overline{|i|^2}}}.$$

With it we get the relations

$$Y_{cor} = Y_{11} - \alpha\sqrt{\frac{\overline{|i_1|^2}}{\overline{|u|^2}}} \text{ or } Z_{cor} = Z_{11} - \alpha\sqrt{\frac{\overline{|u_1|^2}}{\overline{|i|^2}}}. \tag{6a}$$

Therefore, the correlation between i and u and also Y_{cor} (Z_{cor}) can be zero even if $\alpha \neq 0$ and a finite correlation exists between i_1 and i_2 (u_1 and u_2). For i_1 (u_1) uncorrelated to i_2 (u_2) is $\alpha = 0$ and we obtain

$$i_n = i_1 \quad \text{or} \quad u_n = u_1 \tag{5a}$$

and

$$Y_{cor} = Y_{11} \quad \text{or} \quad Z_{cor} = Z_{11}. \tag{6b}$$

Introducing (5) into (2a) we get

$$I_1 = I_1' + i_n + uY_{cor} \qquad I_1 = I_1' + i$$

$$\text{or} \tag{7}$$

$$U_1 = U_1' + u \qquad U_1 = U^1 + u_n + iZ_{cor}$$

and further

$$\begin{aligned} I_1 &= I_1' + i_n + U_1Y_{cor} - U_1'Y_{cor} \\ U_1 &= U_1' + u \\ \text{or} \quad I_1 &= I_1' + i \\ U_1 &= U_1' + u_n + I_1Z_{cor} - I_1'Z_{cor}. \end{aligned} \tag{8}$$

Eq. (8) for the noise fourpole can be realized by the equivalent networks of Fig. 2(b). They only consist of the uncorrelated noise sources u and i_n (u_n and i) and the correlation admittance $+Y_{cor}$ (correlation impedance $+Z_{cor}$) at the input side and the correlation admittance $-Y_{cor}$ (correlation impedance $-Z_{cor}$) at the output side. Both admittances (impedances) are noise-free and have, therefore, the noise temperature $T=0$. Using the well-known Nyquist formulas

$$\overline{|u|^2} = 4kT_0\Delta f\, R_n \qquad \overline{|u_n|^2} = 4kT_0\Delta f\, r_n$$

$$\text{or} \tag{9}$$

$$\overline{|i_n|^2} = 4kT_0\Delta f\, G_n \qquad \overline{|i|^2} = 4kT_0\Delta f\, g_n,$$

we express the noise currents and voltages by the characteristic noise terms:

equivalent noise resistance R_n or r_n,
equivalent noise conductance G_n or g_n.

So we get the equivalent networks of Fig. 2(c) for the noise fourpoles. They describe completely the noise behavior of the whole network by the three terms R_n, G_n, and Y_{cor} (r_n, g_n, and Z_{cor}). As Y_{cor} (Z_{cor}) is complex four real characteristic noise terms are needed in fact.

Between the characteristic noise terms of the Π-matrix and those of the T-matrix the following transformation rules exist similar as they exist for fourpole coefficients

$$g_n = G_n + R_n|Y_{cor}|^2 \qquad R_n = r_n + g_n|Z_{cor}|^2$$

$$r_n = \frac{G_n}{|Y_{cor}|^2 + (G_n/R_n)} \quad \text{or} \quad G_n = \frac{r_n}{|Z_{cor}|^2 + (r_n/g_n)}$$

$$Z_{cor} = \frac{Y_{cor}^*}{|Y_{cor}|^2 + (G_n/R_n)} \qquad Y_{cor} = \frac{Z_{cor}^*}{|Z_{cor}|^2 + (r_n/g_n)}.$$

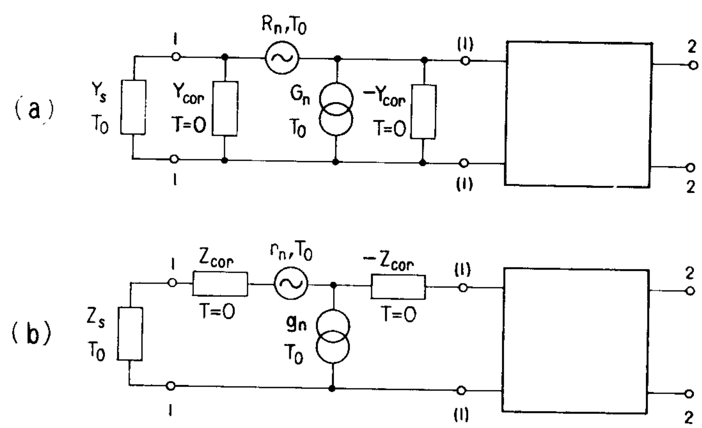

Fig. 3—Equivalent circuit of Fig. 2(c) together with signal source.

The Total Noise Conductance G_{tot}

In the operation of a fourpole a signal source with the inner admittance $Y_s = G_s + jB_s$ (inner impedance $Z = R + jX$) is connected to the input terminals *1 1*, as Fig. 3 shows. In the case of the Π-circuit in Fig. 3(a) the inner conductance G_s of the signal source delivers a noise current inflow i_s in the terminals *1 1* that is uncorrelated to all other noise sources. Therefore, the sum of noise power at the output of the fourpole comes from the signal source as well as from the fourpole. But the whole noise power can be assumed as engendered by a single total equivalent noise current i_{tot} flowing into the input terminals. As all noise sources of the equivalent network of Fig. 3(a) are located at the left side of the terminals *(1)(1)*, the short circuit noise current between *(1)(1)* must be identical with this total equivalent noise current i_{tot}. It is easy calculated to

$$i_{tot} = i_s + i_n + u(Y_s + Y_{cor}). \qquad (11)$$

In (11) each component of i_{tot} is uncorrelated to the other ones. Therefore, the mean square value $\overline{|i_{tot}|^2}$ is equal to the sum of the mean square values of each part

$$\overline{|i_{tot}|^2} = \overline{|i_s|^2} + \overline{|i_n|^2} + \overline{|u|^2}|Y_s + Y_{cor}|^2. \qquad (12)$$

Introducing the total noise conductance G_{tot} by the Nyquist formula

$$\overline{|i_{tot}|^2} = 4kT_0\Delta f G_{tot} \qquad (13)$$

and using (9) we obtain

$$G_{tot} = G_s + G_n + R_n|Y_s + Y_{cor}|^2$$
$$= G_s + G_n + R_n[(G_s + G_{cor})^2 + (B_s + B_{cor})^2]. \qquad (14)$$

and in the case that $G_s \to 0$

$$G_{tot}{}^0 = G_N + R_N[G_{cor}{}^2 + (B_s + B_{cor})^2]. \qquad (14a)$$

This expression for the total noise conductance shows that a complete characterization of the noise quality needs again the four values R_n, G_n, G_{cor}, and B_{cor} besides the admittance Y_s. The total noise conductance $G_{tot}{}^0$ determines the noise behavior of the network in a very simple way. It solely consists of admittances, respectively, conductances and resistances which are independent of the bandwidth of the network. We notice that G_{tot} does not depend on the loading admittance at the output of the fourpole. But it depends on account of the term $R_n|Y_s + Y_{cor}|^2$ upon the real part G_s as well as on the imaginary part jB_s of the source admittance.

The function $G_{tot} = f(B_s)$ is a quadratic parabola, symmetrical to the ordinate $B_s = -B_{cor}$ as shown by Fig. 4. The second differential quotient of the parabola is $2R_n$. The minimum of G_{tot} at the vertex of the parabola is equal to

$$G_{tot\,min} = G_s + G_n + R_n(G_s + G_{cor})^2 \qquad (15)$$

respectively for $G_s \to 0$

$$C_{tot}{}^0{}_{min} = G_n + R_N G_{cor}{}^2. \qquad (15a)$$

Analog relations are valid for the dual T-network, as shown in Fig. 3(b). All noise sources can be replaced by the total equivalent noise resistance

$$R_{tot} = R_s + r_n + g_n|Z_s + Z_{cor}|^2. \qquad (16)$$

Experimental Determination of the Characteristic Noise Values

The experimental methods to determine the noise current sources f.e. using a noise diode are well known. If G_{tot} is measured as function of the source susceptance B_s we find the value of B_{cor} by the tuning condition for $G_{tot\,min}$. If $G_{tot\,min} - G_s$ is plotted as function of G_s we find corresponding to (14) a quadratic parabola with the second differential quotient equal to $2R_n$ as shown in Fig. 5. In the vertex of the parabola we have the ordinate value $G_{tot\,min} - G_s = G_n$ and the abscissa value $G_s = -G_{cor}$. By measuring $G_{tot} = f(B_s)$ and $G_{tot\,min} = f(G_s)$ we therefore find the four values R_n, G_n, G_{cor} and B_{cor}. Because usually no negative values of G_s are available the vertex of the parabola must be found by extrapolation if $G_{cor} > 0$. But on principle positive as well as negative values of G_{cor} and B_{cor} are possible and also measured. The methods to determine the characteristic noise values r_n, g_n, R_{cor} and X_{cor} for the T-circuit are the same on principle.

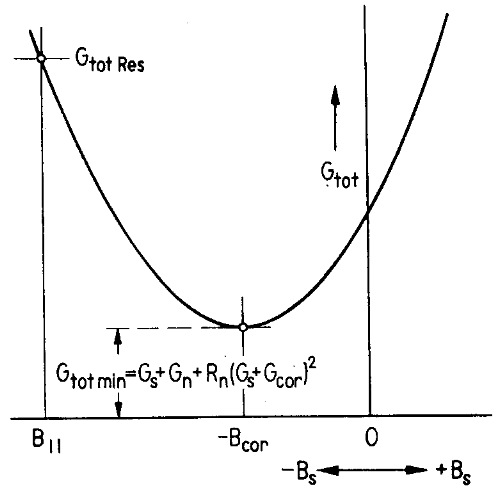

Fig. 4—Total noise admittance G_{tot} as function of the signal source susceptance B_s.

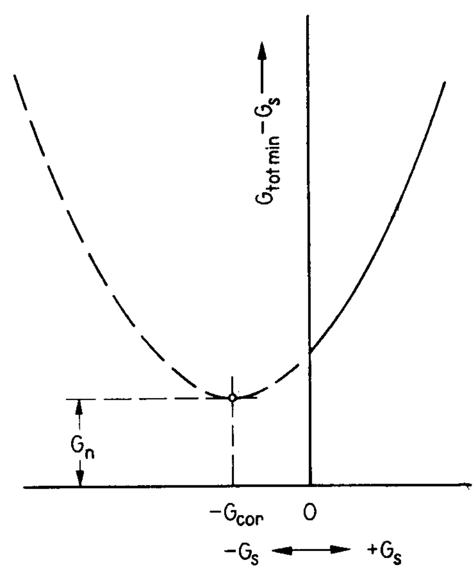

Fig. 5—$G_{tot\,min} - G_s$ as function of the signal source conductance G_s.

Calculation of the Noise Figure

By introducing the total noise conductance G_{tot} (14) into the well-known definition of the excess noise figure [3, 10, 11]

$$F_z = F - 1 = \frac{\overline{|i_{tot}|^2} - \overline{|i_s|^2}}{\overline{|i_s|^2}} = \frac{G_{tot}}{G_s} - 1 \tag{17}$$

we obtain

$$F_z = \frac{1}{G_s}(G_n + R_n|Y_s + Y_{cor}|^2). \tag{18}$$

This function of G_s has its lowest value

$$F_{z\,opt} = 2R_n(G_{cor} + G_{s\,opt}) = 2(R_nG_{cor} + \sqrt{R_nG_{tot}^0})$$
$$= 2[R_nG_{cor} + \sqrt{R_nG_n + (R_nG_{cor})^2 + R_n^2(B_s + B_{cor})^2}] \tag{19}$$

for the optimal internal conductance

$$G_{s\,opt} = \sqrt{\frac{G_n}{R_n} + |jB_s + Y_{cor}|^2} = \sqrt{\frac{G_{tot}^0}{R_n}} \tag{20}$$

of the signal source. This condition is called *noise matching*. As shown by (17) and (18) the values $F_{z\,opt}$ and $G_{s\,opt}$ depend on the tuning of the signal source. If we choose

$$B_s + B_{cor} = 0 \tag{21}$$

which condition is independent of the fourpole's input susceptance B_{11} the excess noise figure becomes its absolute minimum

$$F_{z\,min} = 2R_n(G_{cor} + G_{s\,min}) = 2(R_nG_{cor}\sqrt{R_nG_{tot\,min}^0}$$
$$= 2[R_nG_{cor} + \sqrt{R_nG_n + (R_nG_{cor})^2}] \tag{22}$$

for the corresponding signal source admittance

$$G_{s\,min} = \sqrt{\frac{G_n}{R_n} + G_{cor}^2} = \frac{G_{tot\,min}^0}{R_n} \tag{23}$$

We call the condition (21) *noise tuning*. To get the minimum noise figure $F_{z\,min}$ the conditions (20) for *noise matching* and (21) for *noise tuning* therefore must be fulfilled together. By (22) the noise figure is solely represented by the products of R_nG_n and R_nG_{cor}.

Influence of the Input Admittance Y_{11}

The input admittance Y_{11} composed of several admittances directly located between the terminals *1 1* of the noisy fourpole may be divided into two principal parts. The first one contains the admittances with the noise power uncorrelated to each of the other noise sources, while the second one contains these other admittances being more or less correlated to the inner noise sources of the fourpole. Let us consider f.e. an hf amplifier using a triode in a neutralized common cathode circuit. Then the whole input admittance Y_{11} consists of the admittance $Y_c = G_c + jB_c$ of the resonance circuit between grid and cathode including the cold input admittance of the tube and delivering uncorrelated noise power and further of the electronic input admittance $Y_{el} = G_{el} + j\omega\Delta C_g$ being closely related to the noise of the electron flow inside of the tube.

In the preceding paragraphs both of these principal parts of noise sources were concentrated into a single noise fourpole to get simple expressions for the noise figure. But this procedure has the decisive disadvantage of preventing a separate discussion of the mentioned two parts of noise, so that G_n and Y_{cor} depend on the noise of the input circuit as well as of the inner noise sources f.e. the electron flow, while R_n is only influenced by the latter one.

To get a complete and separate information on the influence of both these parts of noise sources we therefore propose to transfer the admittance Y_c with uncorrelated noise really located inside of the noisy fourpole to the outside of its terminals *1 1* that is parallel to the admittance Y_s of the signal source. To do this outgrouping of circuit noise (see Table I) it is only necessary to introduce into all equations of the sections "The Total Noise Conductance G_{tot}" and "Experimental Determination of the Characteristic Noise Values".

TABLE I

instead of	the new terms
i_s	$i_y = i_s + i_c$
$\overline{\lvert i_s\rvert^2}$	$\overline{\lvert i_Y\rvert^2} = \overline{\lvert i_s\rvert^2} + \overline{\lvert i_c\rvert^2} = 4kT_0\Delta f(G_s + G_c)$
$Y_s = G_s = jB_s$	$Y = G + jB = Y_s + Y_c = (G_s + G_c) + j(B_s + B_c)$
G_s	$G = G_s + G_c$
$G_s \to 0$	$G_s + G_c \to 0$
B_s	$B = B_s + B_c$

It is easy to prove that G_{tot} resp. G_{tot}^0 is unchanged by this transformation while on the other hand the characteristics G_n and Y_{cor} are changed in quantity and physical interpretation [2]. In the above discussed example of an hf amplifier the new terms G_n and Y_{cor} now represent the noise behavior of the electron flow only, while the influence of the resonance circuit including the cold input admittance of the tube is represented by the admittance Y_c.

The Noise Figure with Separated Y_c

To get the influence of Y_c on the noise figure we have to use in (17) the expression of G_{tot} obtained by introducing the new terms given by (24) into (14). So we find

$$F_z = \frac{1}{G_s}(G_c + G_n + R_n|Y_s + Y_c + Y_{cor}|^2) \tag{18a}$$

and therefore for *noise matching*

$$F_{z\,opt} = 2R_n(G_c + G_{cor} + G_{s\,opt})$$
$$= 2[R_n(G_c + G_{cor}) + \sqrt{R_n(G_c + G_n) + R_n^2(G_c + G_{cor})^2 + R_n^2(B_s + B_c + B_{cor})^2}] \tag{19a}$$

and in the case that $G_c \to 0$

$$F_{z\,\mathrm{opt}} = 2(R_nG_{\mathrm{cor}} + \sqrt{R_N G_{\mathrm{tot}}^0})$$
$$= 2\left[R_nG_{\mathrm{cor}} + \sqrt{R_n[G_n + R_nG_{\mathrm{cor}}^2 + R_n(B_s + B_c + B_{\mathrm{cor}})^2}\right], \tag{19b}$$

with the optimal source conductance

$$G_{s\,\mathrm{opt}} = \sqrt{\frac{G_c + G_n}{R_n} + |jB_s + Y_c + Y_{\mathrm{cor}}|^2} \tag{20a}$$

$$G_{s\,\mathrm{opt}} = \sqrt{\frac{G_{\mathrm{tot}}^0}{R_n}} \tag{20b}$$

For *noise tuning* the tuning condition

$$B_s + B_c + B_{\mathrm{cor}} = 0 \tag{21a}$$

is valid. We obtain

$$F_{z\,\mathrm{min}} = 2R_n(G_c + G_{\mathrm{cor}} + G_{s\,\mathrm{min}})$$
$$= 2\left[R_n(G_c + G_{\mathrm{cor}}) + \sqrt{R_n(G_c + G_n) + R_n^2(G_c + G_{\mathrm{cor}})^2}\right] \tag{22a}$$

respectively for $G_c \to 0$

$$F_{z\,\mathrm{min}} = 2(R_nC_{\mathrm{cor}} + \sqrt{R_nG_{\mathrm{tot}}^0{}_{\mathrm{min}}})$$
$$= 2\left[R_nG_{\mathrm{cor}} + \sqrt{R_n(G_n + R_nG_{\mathrm{cor}}^2)}\right] \tag{22b}$$

with

$$G_{s\,\mathrm{min}} = \sqrt{\frac{G_c + G_n}{R_n} + (G_c + G_{\mathrm{cor}})^2}. \tag{23a}$$

with

$$G_{s\,\mathrm{min}} = \frac{\sqrt{G_{\mathrm{tot}}^0{}_{\mathrm{min}}}}{R_n}. \tag{23b}$$

If $G_c \ll G_n$, G_{cor} (22a) and (23a) are identical with (22) and (23). If further $G_{\mathrm{cor}} = 0$ we get

$$F_{z\,\mathrm{min}} \to 2\sqrt{R_nG_n}, \tag{24}$$

$$G_{s\,\mathrm{min}} \to \sqrt{G_n/R_n}. \tag{25}$$

Both expressions only imply the noise terms R_n and G_n. They are valid for triodes in uhf region.

For triodes at low frequencies $G_c \gg G_n$, G_{cor} (22a) and (23a) simplify to

$$F_{z\,\mathrm{min}} \to 2\left[R_nG_c + \sqrt{R_nG_c + (R_nG_c)^2}\right], \tag{26}$$

$$G_{s\,\mathrm{min}} \to \sqrt{G_c^2 + (G_c/R_n)} \tag{27}$$

depending on G_c and R_n only.

To study the excess noise figure as function of the noise matching of the signal source it is useful to transform (18a) into the form of a circle. For that purpose we introduce the expressions $F_{z\,\mathrm{min}}$ and $G_{s\,\mathrm{min}}$ by (22a) and (23a) and receive

$$F_z = F_{z\,\mathrm{min}} + R_nG_{s\,\mathrm{min}}\left(m + \frac{1}{m} - 2\right) \tag{28}$$

with

$$m + \frac{1}{m} = \frac{G_{s\,\mathrm{min}}}{G_s}\left[1 + \left(\frac{G_s}{G_{s\,\mathrm{min}}}\right)^2 + \left(\frac{B_s + B_c + B_{\mathrm{cor}}}{G_{s\,\mathrm{min}}}\right)^2\right]. \tag{29}$$

The coefficient m only depends on the quotients G_s/G_{min}, respectively on

$$\frac{B_s + B_c + B_{\mathrm{cor}}}{G_{s\,\mathrm{min}}}$$

and represents the standing wave ratio $U_{\mathrm{max}}/U_{\mathrm{min}}$ of a transmission line with a wave-resistance $Z = 1/G_{s\,\mathrm{min}}$ being connected to the signal source with the inner admittance $G_s + j(B_s + B_c + B_{\mathrm{cor}})$.

In the complex plane with G_s/G_{min} as abscissa and $(B_s + B_c + B_{\mathrm{cor}})\ G_{s\,\mathrm{min}}$ as ordinate the curves of constant m and therefore also constant noise figure F_z are circles as shown in the well-known matching diagram of Fig. 6. For $G_s = G_{s\,\mathrm{min}}$ and $B_s + B_c + B_{\mathrm{cor}} = 0$ we get $m = 1$ and therefore $F_z = F_{z\,\mathrm{min}}$. The center point of minimum noise figure was already introduced as fulfilling the conditions of noise matching and noise tuning. The points of every circle with extreme values of $B_s + B_c + B_{\mathrm{cor}}$ are fulfilling the condition of noise matching (20a). The locations of this condition are given by the dashed hyperbola

$$\left(\frac{G_{s\,\mathrm{opt}}}{G_{s\,\mathrm{min}}}\right)^2 - \left(\frac{B_s + B_c + B_{\mathrm{cor}}}{G_{s\,\mathrm{min}}}\right)^2 = 1. \tag{30}$$

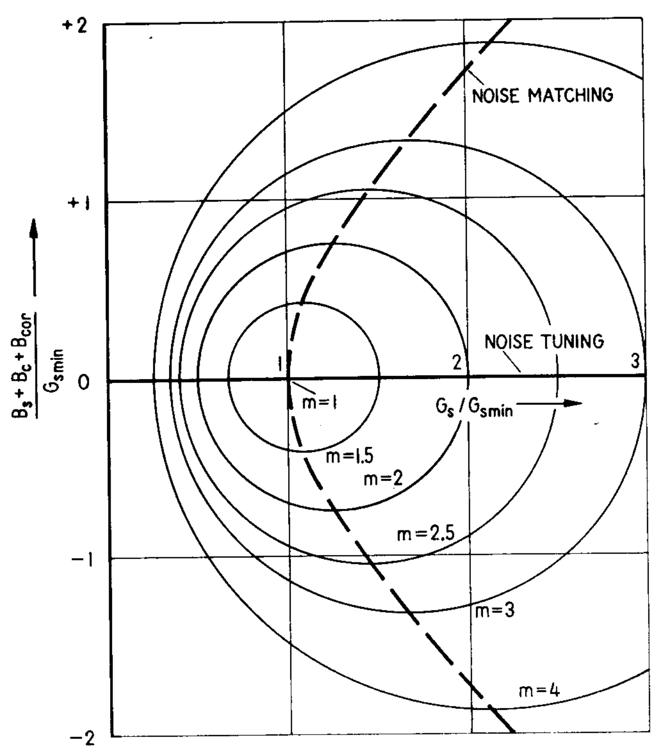

Fig. 6—Noise matching diagram between the signal source and the noise fourpole. The circles are curves for constant noise figure.

Of special interest is the influence of G_{cor} upon the magnitude of $F_{z\,\mathrm{opt}}$ and $F_{z\,\mathrm{min}}$ along this hyperbola for noise matching. In Fig. 7 $F_{z\,\mathrm{opt}}$ corresponding to (19a) is given as function of $B=B_s+B_c$ for a negative value of G_{cor}. This curve is again a quadratic hyperbola symmetrical to $B=-B_{cor}$ and a slope of the asymptotes equal to $\pm R_n$. In the vertex the curve is crossing the center point of Fig. 6 with $F_{z\,opt}=F_{z\,min}$. The point of intersection of the asymptotes has a negative value of $F_{z\,opt}=-2R_n(G_c+G_{cor})$ in our example. The distance

$$2\sqrt{R_n(G_c+G_{cor})+R_n^2(G_c+G_{cor})^2}$$

between this point and the vertex of the hyperbola is always positive and greater than $2R_n(G_c+G_{cor})$, so that $F_{z\,min}$ remains positive. Positive values of G_{cor} shift the hyperbola to higher values of $F_{z\,opt}$, negative G_{cor} to lower values. For $G_c+G_{cor}\to-\infty$ the vertex and, therefore, $F_{z\,min}$ is going to zero.

We have to notice that the condition for *noise tuning* given by (21a) is independent of the impedance $Y_{11}=G_{11}+jB_{11}$ of the fourpole. For *power matching* the conditions

$$Y_s^* = Y_c + Y_{11} \tag{31}$$

or

$$B_s + B_c + B_{cor} = 0$$
$$G_s = G_c + G_{11} \tag{31a}$$

are valid. Therefore, power matching is only identical with noise matching and noise tuning if $G_c+G_{11}=G_{s\,min}$ and $B_{cor}=B_{11}$. In this case the diagram of Fig. 6 is identical with the diagram for power matching. If these conditions are not fulfilled the noise figure for power matching is higher than $F_{z\,min}$ and given by

$$F_z = \frac{1}{G_c+G_{11}}\,[G_c + G_n + R_n(2G_c + G_{11} + G_{cor})^2 + R_n(B_{cor} - B_{11})^2]. \tag{32}$$

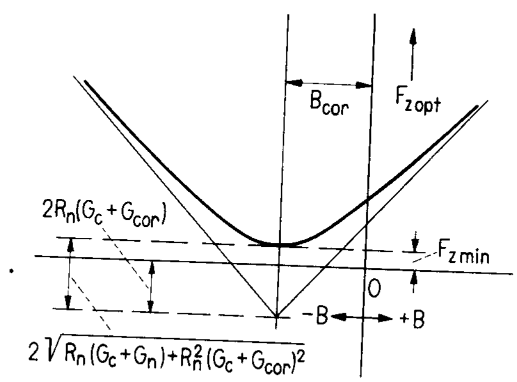

Fig. 7—Noise figure $F_{z\,opt}$ as function of the source susceptance $B=B_s+B_c$. In this example is (G_c+G_{cor}) assumed as negative.

The lowest noise figure in this case is not always attained with $G_c=0$ but sometimes with $G_c>0$ [5]. Using internal feedback inside of the noisy fourpole the conditions for noise matching, noise tuning and power matching can often be combined. Then the absolute minimum value $F_{z\,min}$ of the excess noise figure occurs together with the reflection-free connection to the signal source.

Chain Connection of Noisy Fourpoles

The noise figure for chain connection of two noisy fourpoles can be calculated by the same principal method [7]. The result

$$F_z = F_z^{\mathrm{I}} + \frac{F_z^{\mathrm{II}}}{V_L} \tag{33}$$

is the same as already given by Friis [3]. F_z^{I} and F_z^{II} are the noise figures of the two fourpoles alone and V_L the available power gain of the first fourpole. The total noise figure of n equal fourpoles chained together each of them with the noise figure F_z^{I} is

$$F_z^n = F_z^{\mathrm{I}}\,\frac{1-(1/V_L)^n}{1-(1/V_L)}\,. \tag{34}$$

For $n\to\infty$ then results

$$F_z^\infty = F_z^{\mathrm{I}}\,\frac{V_L}{V_L-1}\,. \tag{35}$$

Fig. 8 shows F_z^n/F_z^{I} as function of V_L for different numbers of n. Eqs. (33) to (35) show very clearly that the noise figure alone is insufficient for full determination of the quality of a noisy fourpole. But the term F_z^∞ given by (35) seems especially adequate as figure of merit.

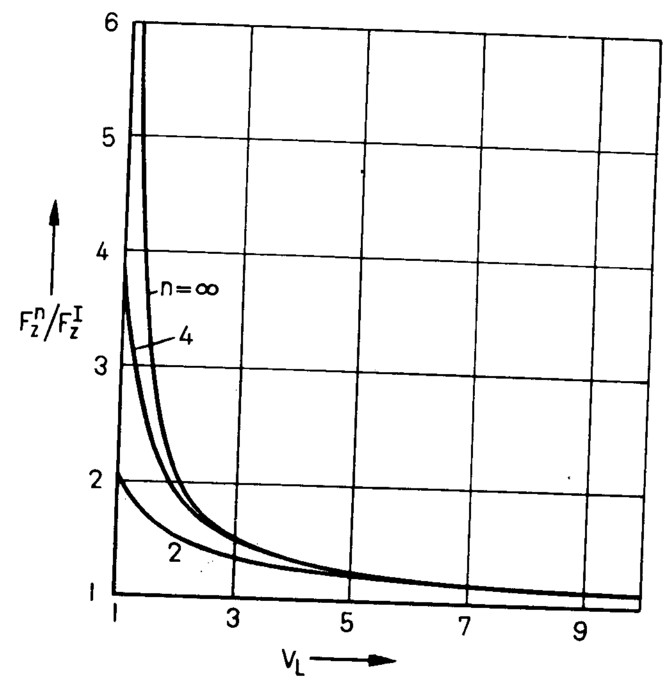

Fig. 8—Relative noise figure of a n-cascaded amplifier as function of the available gain V_L.

Mixer Circuits

The noise properties of mixer stages can also be described by the noise fourpole. Then the short circuit noise currents i_1 and i_2 in Eq. (1) depend on the period of the oscillator frequency ω_o. As i_1 concerns to the input side with the high frequency ω_h and i_2 to the intermediate frequency ω_i in most mixer currents i_1 is practically uncorrelated to i_2. Then we obtain

$$G_n = G_1 \quad (36)$$

with G_1 given by the Nyquist equation

$$\overline{|i_1|^2} = 4kT_0\Delta f G_1$$

and

$$Y_{cor} = Y_{11} \quad (37)$$

where Y_{11} is the mean value of the input admittance for the high frequency ω_h middled over the period of the oscillator frequency ω_o.

Application of the Method

The application of the above considerations on electron tubes was already shown in earlier papers [5, 6, 8]. For triodes at higher frequencies with $C_{ga} = C_{ka} = 0$ respectively in neutralized circuits the short circuit noise current i_i is identical with the induced grid noise current i_g at the input side and i_2 with the space charge suppressed shot noise i_a at the output side. R_n is identical with the well-known equivalent noise resistance R_{eq} and independent of frequency in first approximation. The correlation admittance Y_{cor} as measure of the correlation between the input noise sources i and u is found to be zero in first mostly sufficient approximation while the equivalent noise conductance $G_n > 0$ up to high frequencies in sufficient approximation is proportional to ω^2(ω = angular frequency) [5, 6]. Only for full correlation between i_g and i_a, G_n would be zero. The two parameters R_n and G_n alone fully prescribe the noise behavior of the electron stream in neutralized triodes.[1]

The minimum noise figure is then given by (25) if G_s is negligibly small. We call this lowest possible value the "electronic noise figure" of the triode. As R_n is independent of frequency while G_n is proportional to the square of the frequency it is possible to calculate this electronic noise figure with help of the low frequency value of R_n and the magnitude of G_n measured f.e. in the 100 mc band. So calculated values agree very well with measured values up to very high frequencies [6].

A feedback over C_{ga} decreases the magnitude of R_n but does not change G_n and gives a finite value of Y_{cor} with a negative conductance G_{cor}. Therefore, the noise figure is lowered by this feedback [2, 8]. In screen grid tubes R_n is again identical with R_{eq}. On behalf of the additional partition noise the admittance G_n is larger than in the comparable triode system. G_n is proportional to ω^n where the exponent n starting with the value 2 for low frequencies increases with frequency. Y_{cor} is no more zero but gets a positive conductance G_{cor} proportional to ω^2 and also a positive susceptance B_{cor}.

This method is also very useful in case of transistors [8, 9] and traveling-wave tubes [1]. It does not only prescribe the noise behavior of the amplifying elements but also gives the possibility of conclusions concerning the location and properties of the noise sources inside of their equivalent networks.

[1] This is valid if the circuit admittance $Y_c = G_c + jB_c$ including the "cold" input admittance of the tube are considered as grouped outside of the fourpole parallel to the signal admittance as described in the section "Influence of the Input Admittance Y_{11}." If Y_c remains located inside of the fourpole the noise current i_1 is including the noise current inflow belonging to G_c. Then we get $G_n' = G_n + G_c$ and $Y_{cor}' = Y_{cor} + Y_c$ while $R_n' = R_n$ is unchanged [2]. G_n' and Y_{cor}', therefore, represent no more alone the noise behavior of the electron flow but also the quality of the input circuit.

Bibliography

[1] Bauer, H., and Rothe, H., "Der äquivalente Rauschvierpol als Weilenvierpol." *Archiv der elektrischen Übertragung*, Vol. 10 (1956) in the press.

[2] Dahlke, W., "Transformationsregeln für rauschende Vierpole." *Archiv der elektrischen Übertragung*, Vol. 9 (September, 1955), pp. 391–401.

[3] Friis, H. T., "Noise Figures of Radio Receivers." Proceedings of the IRE, Vol. 32 (July, 1944), pp. 419–422.

[4] Montgomery, H. C., "Transistor Noise in Circuit Applications." Proceedings of the IRE, Vol. 40 (November, 1952), pp. 1461–1471.

[5] Rothe, H., "Die Grenzempfindlichkeit von Verstärkerröhren. Teil III: Äquivalenter Rauschleitwert und Geräuschzahl." *Archiv der elektrischen Übertragung*, Vol. 8 (May, 1954), pp. 201–212.

[6] Rothe, H., "Röhren für Ein- und Ausgangsstufen im 4000-MHz-Gebiet." *Fernmeldetechnische Zeitschrift*, Vol. 7 (October, 1954), pp. 532–539.

[7] Rothe, H., and Dahlke, W., "Theorie rauschender Vierpole." *Archiv der elektrischen Übertragung*, Vol. 9 (March, 1955), pp. 117–121.

[8] Rothe, H., "Die Theorie rauschender Vierpole und ihre Anwendung." *Nachrichtentechnische Fortschritte*, Heft 2 (1955), pp. 24–36.

[9] Schubert, J., "Rauscheigenschaften der Transistoren." Lecture held in the Symposium "Rauschen" ("Noise") of the "Nachrichtentechnische Gesellschaft" in Munich, Germany, (April, 1955).

[10] Standards on Electron Devices: "Methods of Measuring Noise." Proceedings of the IRE, Vol. 41 (July, 1953), pp. 890–896.

[11] van der Ziel, A., "Noise." New York, Prentice-Hall, Inc., 1954

Correction

Joseph E. Rowe, author of the paper, "Design Information on Large-Signal Traveling-Wave Amplifiers," which appeared on pages 200–210 of the February, 1956 issue of Proceedings of the IRE, has informed the editors that as additional calculations were being carried out on the effect on saturation power output and efficiency of loss along the helix of a traveling-wave amplifier, a computer error in the calculations for one curve of the paper was brought to his attention. The error occurred near $y = 4.0$ in the $d = 0.25$ solution of Figs. 14 and 15 on page 206. It is believed that the error originated in the low-order bits of a MIDAC word and then propagated to the higher-order bits as computations continued.