Some Notes on Financial Derivatives for MATH 630 Continued

Tyrone E. Duncan
Mathematics
University of Kansas

Abstract

In these continued notes some discussion of various types of financial derivatives in continuous time is given particularly with an approach to a derivation of the Black-Scholes formula for pricing options in continuous time and some comments about Brownian motion.

1 Brownian Motion-Some History

1836 Scottish botanist Robert Brown viewed the motion of grains of pollen of a plant in water.
1784,1785 Brown's observation had already been reported by others.

Louis Bachelier did calculations with Brownian motion in a 1900 thesis modelling the French stock market (Bourse) with advisor Henri Poincaré. Bachelier did a calculation for a barrier option. He is now considered the father of financial mathematics.
Norbert Wiener in 1923 constructed mathematically the Brownian motion process. He was born in Columbia, MO. He started college (Tufts) at age 10 and received a Ph.D. from Harvard at age 18.
Paul Lévy, a French probabilist, did many explicit calculations for Brownian motion which is also called the Wiener process.
Weierstrass toward the end of the nineteenth century surprised the mathematical world by constructing a continuous function that is nowhere differentiable.

In 1973 F. Black and M. Scholes obtained a partial differential equation whose solution gives the pricing for an option in continuous time.

2 A Stochastic Approach to the Black-Scholes Equation

The stock price S is modeled by a so-called geometric Brownian motion that satisfies the following stochastic differential equation.

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \tag{1}$$

when B is a Brownian motion and μ, σ are constants. X_0 is the initial wealth and $\Delta(t)$ is the amount of shares at time t. Δ can be random but it can only depend on the past price history of the stock. Let $X(t)$ be the wealth at time t. Then X satisfies the followng stochastic differential equation.

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r[X(t) - \Delta(t)]S(t)dt \qquad (2) \\ &= \Delta(t)[\mu S(t)dt + \sigma S(t)dB(t)] + r[X(t) - \Delta(t)S(t)]dt \\ &= rX(t)dt + \Delta(t)S(t)(\mu - r)dt + \Delta(t)S(t)\sigma dB(t) \end{aligned}$$

Let $v(t, S(t))$ be the value of the option at time t.
Consider a European option with value $g(S(T))$ at time T. Let $v(t, x)$ be the value of the option at time t if the stock price $S(t) = x$. This can also be denoted $v(t, S(t))$. Then a stochastic differential equation is described for v.

$$\begin{aligned} dv(t, S(t)) &= v_t dt + v_x dS(t) + \frac{1}{2} v_{xx} dSdS \qquad (3) \\ &= v_t dt + v_x[\mu S(t)dt + \sigma S dB] + \frac{1}{2} v_{xx}\sigma^2 S^2 dt \\ &= [v_t + \mu S v_x + \frac{1}{2}\sigma^2 S^2 v_{xx}]dt + \sigma S v_x dB \end{aligned}$$

The hedging strategy $X(t) = v(t, S(t))$, that is, X tracks v and an equation from above is

$$dX(t) = [rX + \Delta(\mu - r)S]dt + \sigma S \Delta dB \tag{4}$$

To force equality between the last two stochastic differential equations to obtain $X(t) = v(t, S(t))$, the coefficients of the terms with dt and with dB

for the two equations are equated. For the terms with dB the hedging rule is obtained

$$\Delta(t) = v_x(t, S(t)) \tag{5}$$

Equating the coefficients with dt one obtains

$$v_t + \mu S v_x + \frac{1}{2}\sigma^2 v_{xx} = rX + \Delta(\mu - r)S \tag{6}$$

Since $\Delta = v_x$ to make $v = X$ it follows that

$$v_t + \mu S v_x + \frac{1}{2}\sigma^2 v_{xx} = rX + v_x(\mu - r)S \tag{7}$$

which reduces to

$$v_t + rS v_x + \frac{1}{2}\sigma^2 S^2 v_{xx} = rv \tag{8}$$

satisfying the terminal condition $v(T, x) = g(x)$. Note that this equation does not depend explicitly on μ.
This partial differential equation is called the Black-Scholes equation for option pricing.

3 An Explict Solution to the Black-Scholes Equation

The following partial differential equation is called the Black-Scholes equation for the two people who constructed it for pricing continuous time options.
The partial differential equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{9}$$

Recall from above that $v = S$. In this partial differential equation σ describes the stock volatility and r is the interest rate. The interest is compounded continuously.
To solve the Black-Scholes equation some transformations are done to reduce this partial differential equation to the so-called heat equation which can be solved to determine the evolution of heat in a thin rod and whose solution is described by the Normal density function.
The following operations are done to obtain the heat equation.

1. To have the solution evolve in a positive time direction let $\tau = T - t$
2. Define a new variable $x = ln(S)$. This substitution will remove the variable S and S^2 multiplying the various derivatives.
3. A substitution of the form $u = exp(\alpha x + \beta r)V$ will remove unwanted constants and first order terms in x.

Now the three operations are performed on the Black-Scholes equation. Let $\tau = T - t$. The PDE becomes

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 \tau} + rS\frac{\partial V}{\partial \tau} - rV \tag{10}$$

Now let $x = ln(S)$ or equivalently $S = e^x$. Then by the chain rule for differentiation

$$\begin{aligned}\frac{\partial V}{\partial x} &= \frac{\partial V}{\partial x}\frac{dx}{dS} = \frac{1}{S}\frac{\partial V}{\partial x} \\ \frac{\partial^2 V}{\partial S^2} &= \frac{1}{S}\frac{\partial}{\partial x}\frac{1}{S}\frac{\partial V}{\partial S} \\ &= \frac{1}{S^2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{S^2}\frac{\partial V}{\partial x}\end{aligned}$$

because $\frac{1}{S} = e^{-x}$
With these operations the value of a European call option with strike price E and expiration time T is given by

$$C(S,t) = SF(A_+) - Ee^{-r(T-t)}F(A_-) \tag{11}$$

F is the distribution function for the standard normal, denoted $N(0,1)$. Thus

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2} dy \tag{12}$$

and the constants A_+ and A_- are given by

$$\begin{aligned}A_+ &= \frac{log(\frac{S}{E}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ A_- &= \frac{log(\frac{S}{E}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\end{aligned}$$

The partial differential equation becomes

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 S}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} - rV \tag{13}$$

This last equation is of the form

$$\frac{\partial V}{\partial \tau} = A\frac{\partial^2 V}{\partial x^2} + B\frac{\partial V}{\partial x} + CV \tag{14}$$

for some constants A, B, C with $A > 0$.

Lemma 3.1. *Let $V(x,\tau)$ satisfy the equation*

$$\frac{\partial V}{\partial \tau} = A\frac{\partial^2 V}{\partial x^2} + B\frac{\partial V}{\partial x} + CV \tag{15}$$

then the solution is

$$V(x,\tau) = \frac{e^{C\tau}}{\sqrt{4\pi A\tau}} \int_{-\infty}^{\infty} exp(-\frac{1}{2}(\frac{(y - x - B\tau}{\sqrt{2A\tau}})^2)dy \tag{16}$$

Now let $A = \frac{1}{2}\sigma^2$, $B = r - \frac{1}{2}\sigma^2$ and $C = -r$ to obtain

$$V(x,\tau) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \exp -\frac{1}{2}(\frac{y - x - (r - \frac{1}{2}\sigma^2)}{\sigma\sqrt{\tau}})^2 f(y)dy \tag{17}$$

Now apply the initial data. For a call option the payoff is given as

$$\begin{aligned} V(S,T) &= S - E \quad &if \quad S > E \\ V(S,T) &= 0 \quad &if \quad S \le E \end{aligned}$$

By the above change of variable $S = e^x$ so that

$$\begin{aligned} V(S,T) &= S - E \quad &if \quad e^x > E \\ V(S,T) &= 0 \quad &if \quad e^x \le E \end{aligned}$$

The boundary condition is $x = ln(E)$. Thus the solution for the option price is

$$V(x,\tau) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{ln(E)}^{\infty} \exp[-\frac{1}{2}(\frac{y - x - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}})^2](e^y - E)dy \tag{18}$$

Since this expression for V is an explicit calculation from the normal density it can be computed using the standard normal tables.
This statement is explained now. There are two terms in the integral and another change of variables is done for the evaluation.

$$V(x,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\frac{ln(E)-x-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}z^2}(e^{x+(r+\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}z)} - E)dz$$

The second term on the right hand side is equal to $-Ee^{-r(T-t)}F(A_-)$. Using the facts that $\tau = T - t$ and $x = ln(S)$ and the definition of A_-.
In a similar fashion the first term is equal to $SF(A_+)$.

$$V(S,t) = SF(A_+) - Ee^{-r(T-t)}F(A_-) \tag{19}$$

Then this equality is an explicit solution to the option pricing problem.
A more elementary type of option is called a binary option because there are only two possible values for it that are described now.

$$\begin{aligned} V(S,T) &= 1 \quad \textit{if} \quad S > E \\ &= 0 \quad \textit{if} \quad S \leq E \end{aligned}$$

The option price $V(S,t)$ is given by

$$V(S,t) = e^{-r(T-t)}F(d_2) \tag{20}$$

where $d_2 = \frac{ln(\frac{S}{E})+(r-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}$.