Some Notes on Financial Derivatives for MATH 630

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Abstract

In these notes some discussion of various types of financial derivatives especially the binomial pricing model for discrete time is given and subsequently an approach to a derivation of the Black-Scholes formula for pricing options in continuous time.

1 Binomial Pricing Model

This model assumes that prices change in a binomial manner, that is, with one value up and one value down. The initial stock price during the period under study is denoted S_0. At each time step, the stock price either goes up by a factor of u or down by a factor of d. It will be useful to visualize tossing a coin at each time step to determine the next price.

The stock price moves up by a factor of u if the coin has heads (H) and moves down by a factor of d if the coin has tails (T).

A typical sequence of the outcome space Ω will be denoted ω, and ω_k will denote the kth element in the sequence ω. We write $S_k(\omega)$ to denote the stock price at time k (i.e. after k tosses) under the outcome ω. Note that $S_k(\omega)$ depends only on the first k tosses, that is, $\omega_1, ..., \omega_k$.
Each S_k is a random variable defined on the outcome space Ω.

Example 3.1 (Pricing a Call Option) Suppose $u = 2; d = 0.5; r = 25\%$ (interest rate), $S_0 = 50$. (In this and all examples, the interest rate quoted is per unit time, and the stock prices $S_0, S_1, ...$ are indexed by the same time periods). We know that

$$\begin{aligned} S_1(\omega) &= 100 \quad \text{if} \quad S_1(\omega) = H \\ &= 25 \quad \text{if} \quad S_1(\omega) = T \end{aligned}$$

Find the value at time zero of a call option to buy one share of stock at time 1 for 50 (i.e. the strike price is 50).
The value of the call at time 1 is

$$\begin{aligned} V_1(\omega) = (S_1(\omega))^+ &= 50 \quad \text{if} \quad \omega_1 = H \\ &= 0 \quad \text{if} \quad \omega_1 = T \end{aligned}$$

Suppose that the option sells for 20 at $t = 0$. Construct a portfolio as follows
1. Sell 3 options for 20 each. Cash outlay is -60.
2. Buy two shares of stock for 50 each. Cash outlay is 100.
3. Borrow 40. Cash outlay is -40.

This portfolio thus requires no initial investment. For this portfolio, the cash outlay at time 1 is:

Outcomes	$\omega_1 = H$	$\omega_1 = T$
Pay-off option	150	0
Sell stock	200	50
Pay-off debt	50	50
The net result	0	0

The arbitrage pricing theory (APT) value of the option at time 0 is $V_0 = 20$. This price is a "fair" or unbiased price for the option.
The result in this example is a specific example of the "fair price" for the option at $t = 0$. This result is basic for understanding the pricing approach for the binomial model.
Some assumptions for this pricing model:
1. Unlimited short selling of stock.
2. Unlimited borrowing.
3. No transaction costs.
4. Agent is a small investor, i.e., his/her trading does not move the market.

Important Observation: The APT value of the option does not depend on the probabilities of H and T.

2 General One Step Procedure for Arbitrage Pricing Model

Suppose a derivative security pays off the amount V_1 at time 1, where V_1 is an $\mathcal{F}_1$-measurable random variable. This measurability means that V_1 is determined by "information" no more than at $t = 1$. This condition is natural for the pricing problem.
The procedure is as follows:

1. Sell the security for V_0 at time 0. (V_0 is to be determined later).
2. Buy Δ_0 shares of stock at time 0. (Δ_0 is also to be determined later)
3. Invest $V_0 - \Delta_0$ in the money market, at risk-free interest rate r. ($V_0 - \Delta_0$ may be negative).
4. Then the wealth at time 1 is

$$\begin{aligned} X_1 &= \Delta_0 S_1 + (1+r)(V_0 - \Delta_0 S_0) \\ &= (1+r_0)V_0 + \Delta_0(S_1 - (1+r)S_0) \end{aligned}$$

5. Choose V_0 and Δ_0 so that $X_1 = V_1$
   regardless of whether the stock goes up or down.

The last condition above can be expressed by two equations (which is fortunate because there are two unknowns):

$$\begin{aligned} (1+r)V_0 + \Delta_0(S_1(H) - (1+r)S_0) = V_1(H) \\ (1+r)V_0 + \Delta_0(S_1(T) - (1+r)S_0) = V_1(T) \end{aligned} \tag{1}$$

These equations use the fact that if S_k is known then V_k is determined.
Subtract the second equation from the first to obtain

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

Substitute this expression for Δ_0 into (1) to obtain

$$\begin{aligned}(1+r)V_0 &= V_1(H) - \Delta_0(S_1(H) - (1+r)S_0) \\ &= V_1(H) - \frac{V_1(H) - V_1(T)}{(u-d)S_0} \\ &= (u-d)^{-1}[(u-d)V_1(H) - (V_1(H) - V_1(T))(u-1-r)]\end{aligned}$$

It was already assumed that $u > d > 0$. Now it is assumed that $d \leq 1+r \leq u$. If this inequality was not satisfied then there would be an arbitrage possibility.
Define $\tilde{p}, \tilde{q}$ as

$$\begin{aligned}\tilde{p} &= \frac{1+r-d}{u-d} \\ \tilde{q} &= \frac{u-1-r}{u-d}\end{aligned}$$

Thus $\tilde{p} > 0$ and $\tilde{q} > 0$. These two probabilities are special probabilities for the option problem.
The (nonarbitrage) price of the call at $t = 0$ is

$$V_0 = \frac{\tilde{p}V_1(H) - \tilde{q}V_1(T)}{(1+r)} \tag{2}$$

Note that we have defined two different probabilities that determine the price of the call which prevents an arbitrage possibility. It can be called the "fair" price of the option. It is shown that this pricing approach provides the nonarbitrage price for the option.

3 Risk Neutral Probability

To understand the resulting probabilities in the example above a special probability measure is constructed from the coin tossing experiment. Let Ω be the family of outcomes from n tosses of a coin. A probability measure $\widetilde{\mathbb{P}}$ on this outcome space of Ω as

$$\widetilde{\mathbb{P}}(\omega_1, ..., \omega_n) = \tilde{p}^{(number(j:\omega_j=H)}\tilde{q}^{number(j:\omega_j=T)} \tag{3}$$

$\widetilde{\mathbb{P}}$ is called the risk neutral probability measure. The expectation for this probability measure is denoted $\widetilde{\mathbb{E}}$. The above equation (2) can be expressed as

$$V_0 = \widetilde{\mathbb{E}}[\frac{1}{1+r}V_1] \tag{4}$$

Note that this equality has the product of the discount factor $\frac{1}{1+r}$ and the value at $t = 1$.
An important notion in probability is the notion of a martingale which is defined now. It can often be interpreted as the mathematical description of a "fair" game.

Definition 3.1. Let $X_n, n = 1, 2....$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_n$ an increasing sequence of σ-algebras (information). This sequence $(X_n, \mathcal{F}_n, n = 1, 2, ...)$ is said to be a martingale if the following equality is satisfied

$$\mathbb{E}[X_m|\mathcal{F}_n] = X_n \tag{5}$$

where $m > n$.
It can be interpreted in gambling that if X_n is your winnings at time n then the expected winnings at time $m > n$ is your current winnings. This would be a fair game.

Theorem 3.2. *Using the probability measure $\widetilde{\mathbb{P}}$ the discounted stock price sequence $((1+r)^{-k}S_k, \mathcal{F}_k, k = 1, 2, ..., n)$ is a martingale sequence.*

Proof. Apply the definition of martingale as follows:
A simple verification gives the following equality.

$$(1 + r^{-(1+k)})X_{k+1} = (1+r)^{-k}X_k + \Delta_k((1 + r^{-(k+1)}S_{k+1} - (1+r)^{-k}S_k) \tag{6}$$

The *portfolio process* denoted $(\Delta_1, ..., \Delta_n)$ is defined by Δ_k is the stock held in the time interval $[k, k+1)$. It is assumed that Δ_k is $\mathcal{F}_k$ measurable, that is, it is determined by the market information at time $t = k$ so no insider trading.
A portfolio process
i) Let X_0 be the initial, nonrandom wealth.
ii) By induction define the wealth process $(X_k, k = 1, 2, .., n)$ as follows

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k)$$

$$
\begin{aligned}
(1+r^{-(k+1)})S_{k+1} &= (1+r)^{-k}S_k + \Delta_k((1+r)^{-(k+1)}S_{k+1} - (1+k)^{-k}S_k) \\
&= (1+r)X_k + \Delta_k(S_{(k+1)} - (1+r)S_k)
\end{aligned}
$$

iii) Note that each X_k is $\mathcal{F}_k$ measurable.

Theorem 3.3. *Using the probability $\widetilde{\mathbb{P}}$ the discounted, self-financing portfolio process $((1+r)^{-k}X_k, \mathcal{F}_k, k = 1, 2, .., n)$ is a martingale.*

Proof. Use the definition of martingale applied to this portfolio process as is given now.

$$
\begin{aligned}
\widetilde{\mathbb{E}}[(1+r)^{-(k+1)}X_{k+1}|\mathcal{F}_k] = (1+r)^{-k}X_k \\
+\Delta_k\widetilde{\mathbb{E}}[(1+r)^{-(k+1)}S_{k+1}|\mathcal{F}_k] - (1+r)^{-k}\Delta_k S_k \\
= (1+r)^{-k}X_k
\end{aligned}
$$

4 Simple European Derivative

The term European has no geographical significance. This derivative is defined as follows.

Definition 4.1. A simple European derivative security with expiration time m is an $\mathcal{F}_m$ measurable random variable V_m. (Thus V_m is determined by the time m.)

Definition 4.2. A simple European derivative security V_m is *hedgeable* if there there is a constant X_0 and a portfolio $\Delta = (\Delta_1, ..., \Delta_{m-1})$ such that the self-financing value process $(X_1, ..., X_m)$ given above satisfies

$$
X_m(\omega) = V_m(\omega) \qquad for \qquad all \qquad \omega \in \Omega
$$

If this equality is satisfied for $k = 0, 1, ..., m$ then X_k is called the arbitrage pricing theory (APT) value at time k for V_m.

Theorem 4.3. *If a simple European derivative V_m is hedgeable then for each $k \in (0, 1, ..., m-1)$ the APT value at time k of V_m is*

$$V_k = (1+r)^k \widetilde{\mathbb{E}}[(1+r)^{-m} V_m | \mathcal{F}_k](\omega_1, ..., \omega_k) \tag{7}$$

for each $k = 0, 1, ..., m-1$, and it follows that

$$\widetilde{\mathbb{E}}[M_{k+1} | \mathcal{F}_k] = M_k \tag{8}$$

for each $k = 0, 1, ..., m-1$ and furthermore for $k = m$.

Proof. The proof follows from the above results.

The equality in the theorem follows by the martingale property verified above.

If the European security V_m is hedgeable then there is a portfolio process whose self-financing process $(X_0, X_1, ..., X_m)$ satisfies $X_m = V_m$. By definition X_k is the APT value at time k of V_m. Thus it follows that $(X_0, (1+r)X_1, ..., (1+r)^{-m} X_m)$ is a martingale and

$$X_k = (1+r)^k \widetilde{\mathbb{E}}[(1+r)^{-m} V_m | \mathcal{F}_k] \tag{9}$$

The binomial model is said to be "complete", if every simple European derivative can be hedged.
The precise description is given now.

Theorem 4.4. *The binomial model is complete. If V_m is a simple European derivative option then define*

$$V_k(\omega_1, ..., \omega_k) = (1+r)^k \widetilde{\mathbb{E}}[(1+r)^{-m} V_m | \mathcal{F}](\omega_1, ..., \omega_k) \tag{10}$$

$$\Delta_k(\omega_1, ..., \omega_k) = \frac{V_{k+1}(\omega_1, ..., \omega_k, H) - V_{k+1}(\omega_1, ..., \omega_k, T)}{S_{k+1}(\omega_1, ..., \omega_k, H) - S_{k+1}(\omega_1, ..., \omega_k, T)} \tag{11}$$

Commencing with the wealth $V_0 = \widetilde{\mathbb{E}}[(1+r)^{-m} V_m]$, the self-financing value of the portfolio process $\Delta_0, ..., \Delta_{m-1}$ is the process $V_0, ..., V_m$.

Proof. Let $V_0, ..., V_{m-1}$ and $\Delta_0, ..., \Delta_{m-1}$ be defined above. Let $X_0 = V_0$ and then define the method by induction

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k) \tag{12}$$

Show that $X_k = V_k$ for $k = 0, 1, .., m$ is satisfied. The verification proceeds by mathematical induction. It is true for $k = 0$ because by definition $X_0 = V_0$. Now assume by induction that it has been verified for k and then show it is true for $k+1$. Show that

$$X_{k+1}(\omega_1, ..., \omega_k, H) = V_{k+1}(\omega_1, ..., \omega_k, H)$$

and correspondingly the equality with H replaced by T. Only the result for H is verified here because the other is similar.

$$\begin{aligned}
\widetilde{\mathbb{E}}[(1+r)^{k+1}V_{k+1}|\mathcal{F}_k] &= \widetilde{\mathbb{E}}[(1+r)^{-m}V_m|\mathcal{F}_{k+1}]|\mathcal{F}_k] \\
&= \widetilde{\mathbb{E}}[(1+r)^{-m}V_m|\mathcal{F}_k] \\
&= (1+r)^{-k}V_k
\end{aligned}$$

Thus $([(1+r)^{-k}V_k],\ k = 0, 1, ..., n)$ is a martingale. Thus

$$\begin{aligned}
V_k(\omega_1, ..., \omega_k) &= \widetilde{\mathbb{E}}[(1+r)^{-1}V_{k+1}|\mathcal{F}_k](\omega_1, ..., \omega_k) \\
&= \frac{1}{1-r}(pV_{k+1}(\omega_1, ..., \omega_k, H) + qV_{k+1}(\omega_1, ..., \omega_k, T))
\end{aligned}$$

Now

$$\begin{aligned}
X_{k+1}(H) &= \Delta_k S_{k+1}(H) + (1+r)(X_k - \Delta_k S_k) \\
&= \Delta_k S_{k+1}(H) - (1+r)S_k + (1+r)V_k \\
&= (V_{k+1}(H) - V_{k+1}(T))\frac{u-1-r}{u-d} + \tilde{p}V_{k+1}(H) + \tilde{q}V_{k+1}(T) \\
&= (V_{k+1}H - V_{k+1}(T) + \tilde{p}V_{k+1}(H) + \tilde{q}V_{k+1}(T) \\
&= V_{k+1}(H)
\end{aligned}$$

This completes the verification.

In a multiperiod binomial process, the valuation has to proceed iteratively (i.e., starting with the final time period and moving backward in time until the current point in time). This procedure is often called dynamic programming. The portfolios replicating the option are created at each step and valued, providing the values for the option in that time period. The final output from the binomial option pricing model is a statement of the value of the option in terms of the replicating portfolio, composed of shares (option delta) of the underlying asset and risk-free borrowing/lending.

The binomial model provides insight into the determination of option value. The value of an option is not determined by the expected price of the asset but by its current price, which, of course, reflects expectations about the future. This is a direct consequence of arbitrage. If the option value deviates from the value of the replicating portfolio, investors can create an arbitrage position (i.e., one that requires no investment, involves no risk, and delivers positive returns). To illustrate, if the portfolio that replicates the call costs more than the call does in the market, an investor could buy the call, sell the replicating portfolio, and be guaranteed the difference as a profit. The cash flows on the two positions will offset each other, leading to no cash flows in subsequent periods. The call option value also increases as the time to expiration is extended, as the price movements (u and d) increase, and with increases in the interest rate.

5 American Options

For an American option the holder can exercise the option in discrete time for say $n = 1, 2, ..., N$. Thus the option holder has to decide when to exercise the option.
Consider the binomial pricing model with n periods. Let $v_n = g(S_n)$ be the payoff of the derivative security.

$$v_k(x) = \frac{1}{r+1}[\tilde{p}v_{k+1}(ux) + \tilde{q}v_{k+1}(dx)] \tag{13}$$

Thus $v_k(S_k)$ is the value of the option at time k, and the hedging portfolio is given by

$$\Delta_k = \frac{v_{k+1}(uS_k) - v_{k+1}(dS_k)}{(u-d)S_k} \tag{14}$$

In any period k the holder of the derivative can exercise and receive payment $g(S_k)$. Thus the hedging portfolio should create a wealth process

$$X_k \geq g(S_k) \qquad \text{for all} \qquad k \tag{15}$$

Thus the American option algorithm is

$$\begin{aligned} v_n(x) &= g(x) \\ v_k(x) &= \max[\frac{1}{r+1}(v_{k+1}(uS_k) - v_{k+1}(dS_k)), g(x)] \end{aligned}$$

Then $V_k(S_k)$ is the value of the option at time k.

Now consider an American put option with strike price 5. The option has a maximum time of two steps.
Initial data: $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, p = q = \frac{1}{2}, n = 2$
Let $v_2(x) = g(x) = (5 - x)^+$
Compute the hedging portfolio for this option. Begin with wealth $X_0 = 1.36$ and compute Δ_0 as follows.

$$\begin{aligned} 0.40 &= v_1(S_1(H)) \\ &= S_1(H)\Delta_0 + (1+r)(X_0 - \Delta_0(X_0 - \Delta_o S_0)) \\ &= 8\Delta_0 + \frac{5}{4}(1.36 - 4\Delta_0) \\ &= 3\Delta_0 + 1.70 \qquad \text{implies} \qquad \Delta_0 = -.43 \end{aligned}$$

$$\begin{aligned} 3.00 &= v_1(S_1(T)) = S_1(T)\Delta_0 + (1+r)(X_0 - \Delta_0 S_0) \\ &= 2\Delta_0 + \frac{5}{4}(1.36 - 4\Delta_0) \\ &= -3\Delta_0 + 1.70 \qquad \text{implies} \qquad \Delta_0 = -.43 \end{aligned}$$

Use $\Delta_0 = -.43$ to obtain

$$\begin{aligned} X_1(H) &= v_1(S_1(H)) = .40 \\ X_1(T) &= v_1(S_1(T)) = 3.00 \end{aligned}$$

Recall that $S_1(H) = 2$.

$$
\begin{aligned}
1 &= v_2(4) \\
&= S_2(TH)\Delta_1(T) + (1+r)(X_1(T) - \Delta(T)S_1(T)) \\
&= 4\Delta_1(T) + \frac{5}{4}(3 - 2\Delta_1(T)) \\
&= 1.5\Delta_1(T) + 3.25 \qquad \text{implies} \qquad \Delta_1(T) = -1.83 \\
4 &= v_2(1) \\
&= S_2(TT)\Delta_1(T) + (1+r)(X_1(T) - \Delta_1(T)S_1(T)) \\
&= \Delta_1(T) + \frac{5}{4}(3 - 2\Delta_1(T)) \\
&= -1.5\Delta_1(T) + 3.75 \qquad \text{implies} \qquad \Delta_1(T) = -.16
\end{aligned}
$$

Note there are different answers for $\Delta_1(T)$. If $X_1(t) = 2$ then the price of a European put option would be

$$
\begin{aligned}
1 &= 1.5\Delta_1(T) + 2.5 \qquad \text{implies} \qquad \Delta_1(T) = -1 \\
4 &= -1.5\Delta_1(T) + 2.5 \qquad \text{implies} \qquad \Delta_1(T) = -1
\end{aligned}
$$