Bivariate Gaussian (Normal) pdf

- Bivariate Gaussian (Normal) pdf

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho_{XY}^2)}}\exp\left(-\frac{1}{2(1-\rho_{XY}^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}-2\rho_{XY}\frac{(x-\mu_X)}{\sigma_X}\frac{(y-\mu_Y)}{\sigma_Y}\right]\right)$$

- The marginals and conditionals are:

$$f_X \propto N(\mu_X,\sigma_X^2); \quad f_{Y|X=x} \propto N(\mu_Y+\rho_{XY}\frac{\sigma_Y}{\sigma_X}(x-\mu_X), \quad \sigma_Y^2(1-\rho_{XY}^2))$$

$$f_Y \propto N(\mu_Y,\sigma_Y^2); \quad \boxed{f_{X|Y=y} \propto N(\mu_X+\rho_{XY}\frac{\sigma_X}{\sigma_Y}(y-\mu_Y), \quad \sigma_X^2(1-\rho_{XY}^2))}$$

Note: If *X* and *Y* are bivariate Gaussian, then $\rho_{XY}= 0$ (Uncorrelated) → independence
ie. When $\rho_{XY} = 0$, $f_{X,Y}(x,y)=f_X(x)f_Y(y)$; This applies **only** to jointly Gaussian variables.

In other cases, Independence → uncorrelated, but
Uncorrelated **does not** imply independence

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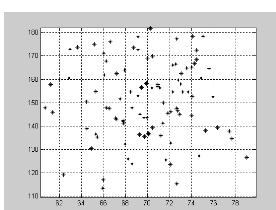
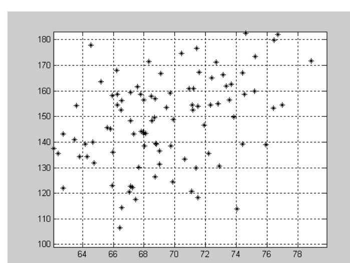
Examples of Samples of Correlated Variables (Jointly Normal)

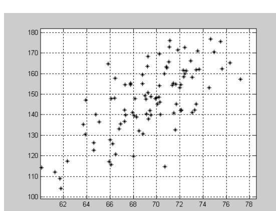
ρ = 0.0

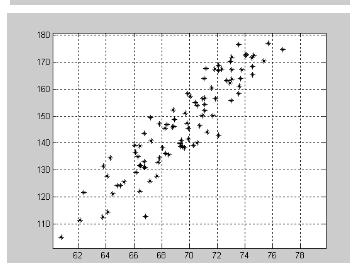




ρ = 0.25

ρ = 0.75





ρ = 0.90

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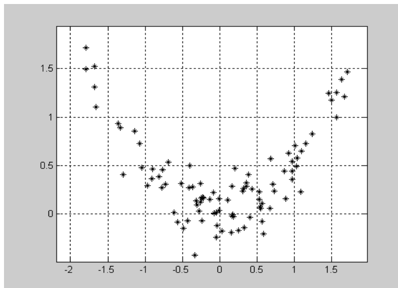
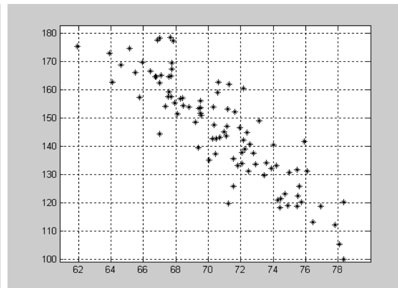
Examples of Samples of Correlated Variables





Not Linearly Correlated
(Non linear correlation)

Negative Linear Correlation

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3

More than Two Random Variables: Random Vectors ; Multivariate Gaussian

- The notion of multiple random variables can be extended beyond to *n > 2* using the vector notation
- One of the important multi dimensional pdf is the multivariate Gaussian

$$\overline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix};\ \overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix};\ f_{\overline{X}}(\overline{x}) = \frac{1}{\sqrt{(2\pi)^n \|\Sigma_{\overline{X}}\|}} \exp\left[-\frac{1}{2} (\overline{x} - \mu_{\overline{X}})^T \Sigma_{\overline{X}}^{-1} (\overline{x} - \mu_{\overline{X}}) \right];$$

Magnitude of the determinant

$$E\{\overline{X}\} = \mu_{\overline{X}} = \begin{bmatrix} \mu_{X_1} \\ \mu_{X_2} \\ \vdots \\ \mu_{X_n} \end{bmatrix};\quad E\{(\overline{X} - \mu_{\overline{X}})(\overline{X} - \mu_{\overline{X}})^T\} = \Sigma_{\overline{X}} = \begin{bmatrix} \sigma_{X_1X_2} & \sigma_{X_1X_2} & \cdots & \sigma_{X_1X_n} \\ \sigma_{X_2X_1} & \sigma_{X_2X_2} & & \sigma_{X_2X_n} \\ \vdots & & \ddots & \vdots \\ \sigma_{X_nX_1} & \sigma_{X_nX_2} & \cdots & \sigma_{X_nX_n} \end{bmatrix}$$

$\mu_{\overline{X}}$ is the mean vector, and $\Sigma_{\overline{X}}$ is the covariance matrix

$$\sigma_{X_iX_j} = E\{(X_i - \mu_i)(X_j - \mu_j)\};\quad \sigma_{X_iX_i} = E\{(X_i - \mu_i)^2\} = \sigma_{X_i}^2$$

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4

More than Two Random Variables: Random Vectors ; Multivariate Gaussian

- All marginal and conditionals are also Gaussian
- *If* $\overline{X}_1$ *and* $\overline{X}_2$ are subvectors of $\overline{X}$, then the condititonal of $\overline{X}_1$ given $\overline{X}_2 = \overline{x}_2$ is Gaussian *with* $\mu_{\overline{X}_1|\overline{X}_2} = \mu_{\overline{X}_1} + \sum_{12}\sum_{22}^{-1}(\overline{x}_2 - \mu_{\overline{X}_2})$, *and* $\sum_{\overline{X}_1|\overline{X}_2} = \sum_{11} - \sum_{12}\sum_{22}^{-1}\sum_{21}$

  where $\sum_{\overline{X}} = \begin{bmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{bmatrix}$ *is a partiton of* $\sum_{\overline{X}}$

Example:

$$\mu_{\overline{X}} = \begin{bmatrix} \mu_{\overline{X}_1} \\ \mu_{\overline{X}_2} \end{bmatrix} = \begin{bmatrix} \mu_{X_1} \\ \mu_{X_2} \\ \mu_{X_3} \\ \mu_{X_3} \end{bmatrix}; \quad \Sigma_{\overline{X}} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \left[\begin{array}{cc:cc} \sigma_{X_1X_2} & \sigma_{X_1X_2} & \sigma_{X_1X_2} & \sigma_{X_1X_n} \\ \sigma_{X_2X_1} & \sigma_{X_2X_2} & \sigma_{X_2X_4} & \sigma_{X_3X_4} \\ \hdashline \sigma_{X_3X_1} & \sigma_{X_3X_2} & \sigma_{X_3X_3} & \sigma_{X_3X_4} \\ \sigma_{X_4X_1} & \sigma_{X_4X_2} & \sigma_{X_4X_3} & \sigma_{X_4X_4} \end{array}\right]$$

If $\overline{Y} = A\overline{X}$, *then* $\overline{Y}$ *is Gaussian with* $\mu_{\overline{Y}} = A\mu_{\overline{X}}$ *and* $\sum_{\overline{Y}} = A \sum_{\overline{X}} A^T$

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5