Poisson Process and the Exponential pdf

N(t) is a point process that can represent the state of the system at time t.

Goal: Find Prob [the system is in state k at t sec] = P(N(t) = k) = P[k, t]
    (if each increment in the process represents an arrival or "birth", then P[k, t] = Probability of # arrivals in t sec)

Analysis

Pure Birth (Poisson) Process: Assumptions

Prob [1 arrivals in Δ t sec] = λ Δ t
Prob [0 arrivals in Δ t sec] = 1 - λ Δ t

Independent Increments

Number of arrivals in non-overlapping intervals of times are statistically independent random variables, i.e.,
Prob [N arrivals in t, t+T AND M arrivals in t+T, t+T+τ]
= Prob [N arrivals in t, t+T]*Prob[M arrivals in t+T, t+T+τ]

This is called a Poisson process or pure birth process
Analysis

Define probability of k in the system at time t = Prob[k, t]

Probability of k in the system at time t+ Δ t = Prob[k, t+ Δ t ]

= Prob[k, t+ Δ t] Prob[(k in the system at time t and 0 arrivals in Δ t)
or (k-1 in the system at time t and 1 arrival in Δ t)]

= (1 - λ Δ t ) Prob[k,t] + λ Δ t Prob[k-1,t]
Analysis

Rearranging terms

\[
\frac{\text{Prob}[k, t+ \Delta t] - \text{Prob}[k,t]}{\Delta t} + \lambda \text{Prob}[k,t] = \lambda \text{Prob}[k-1,t]
\]

Letting \( \Delta t \rightarrow 0 \) results in the following differential equation:

\[
\frac{d\text{Prob}[k,t]}{dt} + \lambda \text{Prob}[k,t] = \lambda \text{Prob}[k-1,t]
\]

Analysis

For \( k = 0 \) the solution is:
- \( \text{Prob}[0,t] = e^{-\lambda t} \)

For \( k = 1 \) the solution is:
- \( \text{Prob}[1,t] = \lambda t e^{-\lambda t} \)

For \( k = 2 \) the solution is:
- \( \text{Prob}[2,t] = \frac{(\lambda t)^2 e^{-\lambda t}}{2} \)
Analysis

In general the solution is a Poisson probability mass function of the form:

\[ \text{Prob} [k, t] = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \]

A Poisson pmf of this form has the following moments:

\[ E[k] = \lambda t \]
\[ \text{Var}[k] = \lambda t \]

Poisson Arrival Process
The number of arrivals in any t second interval follows a Poisson probability mass function.
Interarrival Time Analysis

Let $\Delta t \to 0$ results in the following

$$P[T_a < t] = \int_0^t f_{T_a}(t) dt = \int_0^t \lambda e^{-\lambda t} dt$$

so

$$f_{T_a}(t) = \lambda e^{-\lambda t} \quad \text{for } t > 0$$

$$f_{T_a}(t) = 0 \quad \text{for } t < 0$$

The distribution of interarrival times is exponential
Interarrival Time Analysis

The interarrival time for a Poisson arrival process follows an exponential probability density function with

\[ E[T_a] = \frac{1}{\lambda} \quad Var[T_a] = \frac{1}{\lambda^2} \]