$$\mathbf{P} = \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$a^{(0)} = [1, 0, 0]$$

| $n$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| $a_0^{(n)}$ | 1 | 0 | 0.250 | 0.187 | 0.203 | | 0.20 |
| $a_1^{(n)}$ | 0 | 0.75 | 0.062 | 0.359 | 0.254 | | 0.28 |
| $a_2^{(n)}$ | 0 | 0.25 | 0.688 | 0.454 | 0.543 | | 0.52 |

$$a^{(0)} = [0, 1, 0]$$

| $n$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| $a_0^{(n)}$ | 0 | 0.25 | 0.187 | 0.203 | 0.199 | | 0.20 |
| $a_1^{(n)}$ | 1 | 0 | 0.375 | 0.250 | 0.289 | | 0.28 |
| $a_2^{(n)}$ | 0 | 0.75 | 0.438 | 0.547 | 0.512 | | 0.52 |

$$a^{(0)} = [0, 0, 1]$$

| $n$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| $a_0^{(n)}$ | 0 | 0.25 | 0.187 | 0.203 | 0.199 | | 0.20 |
| $a_1^{(n)}$ | 0 | 0.25 | 0.313 | 0.266 | 0.285 | | 0.28 |
| $a_2^{(n)}$ | 1 | 0.50 | 0.500 | 0.531 | 0.516 | | 0.52 |

**Table I.1**
*Some Properties of the z-Transform*

| SEQUENCE | | z-TRANSFORM |
|---|---|---|
| 1. $f_n \quad n = 0, 1, 2, \ldots$ | $\Leftrightarrow$ | $F(z) = \sum_{n=0}^{\infty} f_n z^n$ |
| 2. $af_n + bg_n$ | | $aF(z) + bG(z)$ |
| 3. $a^n f_n$ | | $F(az)$ |
| 4. $f_{n/k} \quad n = 0, k, 2k, \ldots$ | | $F(z^k)$ |
| 5. $f_{n+1}$ | | $\frac{1}{z}[F(z) - f_0]$ |
| 6. $f_{n+k} \quad k > 0$ | | $\frac{F(z)}{z^k} - \sum_{i=1}^{k} z^{i-k-1} f_{i-1}$ |
| 7. $f_{n-1}$ | | $zF(z)$ |
| 8. $f_{n-k} \quad k > 0$ | | $z^k F(z)$ |
| 9. $nf_n$ | | $z \frac{d}{dz} F(z)$ |
| 10. $n(n-1)(n-2), \ldots, (n-m+1)f_n$ | | $z^m \frac{d^m}{dz^m} F(z)$ |
| 11. $f_n \circledast g_n$ | | $F(z)G(z)$ |
| 12. $f_n - f_{n-1}$ | | $(1-z)F(z)$ |
| 13. $\sum_{k=0}^{n} f_k \quad n = 0, 1, 2, \ldots$ | | $\frac{F(z)}{1-z}$ |
| 14. $\frac{\partial}{\partial a} f_n$ ($a$ is a parameter of $f_n$) | | $\frac{\partial}{\partial a} F(z)$ |
| 15. Series sum property | | $F(1) = \sum_{n=0}^{\infty} f_n$ |
| 16. Alternating sum property | | $F(-1) = \sum_{n=0}^{\infty} (-1)^n f_n$ |
| 17. Initial value theorem | | $F(0) = f_0$ |
| 18. Intermediate value theorem | | $\left.\frac{1}{n!} \frac{d^n F(z)}{dz^n}\right\|_{z=0} = f_n$ |
| 19. Final value theorem | | $\lim_{z \to 1} (1-z)F(z) = f_\infty$ |

*Some z-Transform Pairs*

| SEQUENCE | | z-TRANSFORM |
|---|---|---|
| 1. $f_n \quad n = 0, 1, 2, \ldots$ | $\Leftrightarrow$ | $F(z) = \sum_{n=0}^{\infty} f_n z^n$ |
| 2. $u_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ | | $1$ |
| 3. $u_{n-k}$ | | $z^k$ |
| 4. $\delta_n = 1 \quad n = 0, 1, 2, \ldots$ | | $\frac{1}{1-z}$ |
| 5. $\delta_{n-k}$ | | $\frac{z^k}{1-z}$ |
| 6. $A\alpha^n$ | | $\frac{A}{1-\alpha z}$ |
| 7. $n\alpha^n$ | | $\frac{\alpha z}{(1-\alpha z)^2}$ |
| 8. $n$ | | $\frac{z}{(1-z)^2}$ |
| 9. $n^2\alpha^n$ | | $\frac{\alpha z(1+\alpha z)}{(1-\alpha z)^3}$ |
| 10. $n^2$ | | $\frac{z(1+z)}{(1-z)^3}$ |
| 11. $(n+1)\alpha^n$ | | $\frac{1}{(1-\alpha z)^2}$ |
| 12. $(n+1)$ | | $\frac{1}{(1-z)^2}$ |
| 13. $\frac{1}{m!}(n+m)(n+m-1)\cdots(n+1)\alpha^n$ | | $\frac{1}{(1-\alpha z)^{m+1}}$ |

EXAMPLE

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

$$I - zP = \begin{bmatrix} 1 - \frac{1}{2}z & -\frac{1}{2}z \\ -\frac{1}{4}z & 1 - \frac{3}{4}z \end{bmatrix}$$

$$|I - zP| = (1 - \tfrac{1}{2}z)(1 - \tfrac{3}{4}z) - \tfrac{1}{8}z^2$$

$$= 1 - \frac{5}{4}z + \frac{1}{4}z^2$$

$$= (1 - z)(1 - \tfrac{1}{4}z)$$

thus

$$[I - zP]^{-1} = \begin{bmatrix} \dfrac{1 - \frac{3}{4}z}{(1-z)(1-\frac{1}{4}z)} & \dfrac{\frac{1}{2}z}{(1-z)(1-\frac{1}{4}z)} \\[2ex] \dfrac{\frac{1}{4}z}{(1-z)(1-\frac{1}{4}z)} & \dfrac{1 - \frac{1}{2}z}{(1-z)(1-\frac{1}{4}z)} \end{bmatrix}$$

we always have pole at z=1 to get this

$$[I - zP]^{-1} = \begin{bmatrix} \frac{1/3}{1-z} + \frac{2/3}{1-\frac{1}{4}z} & \frac{2/3}{1-z} + \frac{-2/3}{1-\frac{1}{4}z} \\ \frac{1/3}{1-z} + \frac{-1/3}{1-\frac{1}{4}z} & \frac{2/3}{1-z} + \frac{1/3}{1-\frac{1}{4}z} \end{bmatrix}$$

$$= \frac{1}{1-z}\begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix} + \frac{1}{1-\frac{1}{4}z}\begin{bmatrix} 2/3 & -2/3 \\ -1/3 & 1/3 \end{bmatrix}$$

TAKE THE INVERSE Z-TRANSFORM TO GET $P^n$

$$P^n = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix} + \left(\frac{1}{4}\right)^n \begin{bmatrix} 2/3 & -2/3 \\ -1/3 & 1/3 \end{bmatrix}$$

$$\boxed{\underline{\pi}^T(n) = \underline{\pi}^T(0)\, P^n}$$

CHECK

$$\Pi^1(0) = (1, 0)$$

$$\Pi^1(1) = \left( \frac{1}{3} + \frac{1}{4}\left(\frac{2}{3}\right),\ \frac{2}{3} - \frac{1}{4}\left(\frac{2}{3}\right) \right)$$

$$= ( .5 \quad , \quad .5 )$$

$$\Pi^1(2) = \left( \frac{1}{3} + \frac{1}{16}\left(\frac{2}{3}\right),\ \frac{2}{3} - \frac{1}{16}\left(\frac{2}{3}\right) \right)$$

$$= ( .375 \quad , \quad .625 )$$

$$\Pi^1(3) = \left( \frac{1}{3} + \frac{1}{64}\left(\frac{2}{3}\right),\ \frac{2}{3} - \frac{1}{64}\left(\frac{2}{3}\right) \right)$$

$$= ( .344 \quad , \quad .656 )$$

etc. ! !

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$[\mathbf{I} - z\mathbf{P}]^{-1} = \frac{1}{(1-z)[1+(1/4)z]^2} \times \begin{bmatrix} 1 - \frac{1}{2}z - \frac{3}{16}z^2 & \frac{3}{4}z - \frac{5}{16}z^2 & \frac{1}{4}z + \frac{9}{16}z^2 \\ \frac{1}{4}z + \frac{1}{16}z^2 & 1 - \frac{1}{2}z - \frac{1}{16}z^2 & \frac{3}{4}z + \frac{1}{16}z^2 \\ \frac{1}{4}z + \frac{1}{16}z^2 & \frac{1}{4}z + \frac{3}{16}z^2 & 1 - \frac{3}{16}z^2 \end{bmatrix}$$

$$\mathbf{P}^n = \frac{1}{25}\begin{bmatrix} 5 & 7 & 13 \\ 5 & 7 & 13 \\ 5 & 7 & 13 \end{bmatrix} + \frac{1}{5}(n+1)\left(-\frac{1}{4}\right)^n \begin{bmatrix} 0 & -8 & 8 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{bmatrix} + \frac{1}{25}\left(-\frac{1}{4}\right)^n \begin{bmatrix} 20 & 33 & -53 \\ -5 & 8 & -3 \\ -5 & -17 & 22 \end{bmatrix} \qquad n = 0, 1, 2, \ldots \quad (2.65)$$