Homework 3, solution

1. (a) Polarization mode dispersion of a standard single mode fiber is specified by its mean unit-length differential group delay $DGD_{\text{mean}}$. For $DGD_{\text{mean}} = 0.06 \text{ ps} / \sqrt{\text{km}}$, what is the average differential delay $\Delta \tau_g$ after 1000km of fiber?

(b) A polarization-maintaining (PM) fiber is highly birefringent, and its birefringence is specified by the beat-length. For a PM fiber with 5mm beat length at 1550nm wavelength, what is the differential index $|n_x - n_y|$?

Solution,

(a) Since $\Delta \tau_g = (\Delta n_{\text{eff}} / c) \sqrt{L} = 0.06 \times \sqrt{100} = 1.9 \text{ ps}$

(b) Based on Eq. (2.5.30), the relative phase change between the two orthogonal polarization modes is $\Delta \Phi = (\omega / c) |n_x - n_y| L = (2\pi / \lambda) |n_x - n_y| L$. For $\Delta \Phi = 2\pi$, the fiber length is equal to the beat-length: $L = L_p$.

$|n_x - n_y| = \frac{2\pi}{(2\pi / \lambda) L_p} = \frac{\lambda}{L_p} = \frac{1550 \times 10^{-9}}{5 \times 10^{-3}} = 3.1 \times 10^{-4}$

2. Single mode fiber has loss parameter $\alpha_{\text{dB}} = 0.23 \text{ dB/km}$ at 1550nm wavelength. Based on Eq. (2.6.14), please plot the nonlinear effective length $L_{\text{eff}}$, as the function of the actual fiber length $L$ from 1km to 100km. Discuss at what fiber length the simplified formula, $L_{\text{eff}} \approx 1/\alpha$ has less than 10% error.

Solution:

$\alpha_{\text{dB}} = 0.23 \text{ dB/km}$ has to be first converted into Neper/km, which is $\alpha = 0.23/4.343 = 0.053 \text{ Np/km}$. $L_{\text{eff}} = (1 - e^{-\alpha L}) / \alpha \approx 1/\alpha$

$1/\alpha = 18.8679 \text{ km}$ which is equal to the effective length $L_{\text{eff}}$ when the fiber length approaches infinity. For 1% error, $e^{-\alpha L} = 0.1$, so that

$L = -\ln(0.1) / \alpha = 43.4 \text{ km}$

That is, when the fiber length is longer than 43.4km, $L_{\text{eff}} \approx 1/\alpha$ is accurate with <10% error.
3. A dispersion-shifted fiber has the effective cross section area \( A_{\text{eff}} = 52 \mu m^2 \), a nonlinear index \( n_2 = 2.2 \times 10^{-20} m^2/W \), and a loss parameter \( \alpha_{\text{dB}} = 0.23 \text{dB/km} \) at 1550nm wavelength. If the fiber length is 5km and the optical signal is continuous wave (CW), what is the optical power \( P \) required at the fiber input to produce nonlinear phase shift of \( \Phi_{\text{NL}} = 1 \text{ radians} \)? What are the powers required if the fiber length are 100km and 500km (to produce \( \Phi_{\text{NL}} = 1 \text{ radians} \))?

Solution:

Based on Eq.(2.6.13), \( \Phi_{\text{NL}} = \gamma P(0) \int_0^L e^{-\alpha z} dz = \gamma P(0) \frac{1-e^{-\alpha L}}{\alpha} \)

Here \( \gamma = \frac{n_2 \omega_0}{c A_{\text{eff}}} = \frac{2m_2 \omega_2^2}{\lambda A_{\text{eff}}} = \frac{2 \pi \times 2.2 \times 10^{-20}}{1550 \times 10^{-9} \times 52 \times 10^{-12}} = 1.7 \times 10^{-3} W^{-1}. \)

\( \alpha = 0.23/4.343 = 0.053 \text{Np/km} = 5.3 \times 10^{-5} \text{Np/m}, \) and

\( \exp(-\alpha L) = \exp(-0.053 \times 5) = 0.7672 \)

For \( \Phi_{\text{NL}} = 1 \text{ radians}, P(0) = \frac{1}{\gamma (1-e^{-\alpha L})} = \frac{5.3 \times 10^{-5}}{1.7 \times 10^{-3} (1-0.7672)} = 134 mW \)

For \( L = 100 \text{km and 500km}, \Phi_{\text{NL}} = \gamma P(0)(1-e^{-\alpha L})/\alpha \approx \gamma P(0)/\alpha \)

so that \( P(0) = \frac{\Phi_{\text{NL}} \alpha}{\gamma} = \frac{5.3 \times 10^{-5}}{1.7 \times 10^{-3}} = 31.2 mW \)

4. A trapezoid-shaped optical pulse shown in the following figure is injected into an optical fiber with the peak power \( P_0 = 300 mW \), pulse width \( T_0 = 1 \text{ns} \), and equal length of leading edge and trailing edge of 0.1ns \((t_L = t_T = 0.1 \text{ns})\). A standard single mode fiber is used with \( A_{\text{eff}} = 80 \mu m^2 \), a nonlinear index \( n_2 = 2.2 \times 10^{-20} m^2/W \), and a loss parameter \( \alpha_{\text{dB}} = 0.23 \text{dB/km} \) at 1550nm wavelength. The fiber length is 80km.

(a) Draw and label the waveforms of nonlinear phase shift, \( \Phi_{\text{NL}}(t) \), and optical frequency shift \( \delta\nu(t) \) at the fiber output

(b) If the fiber chromatic dispersion is 16ps/(nm-km), please estimate the arrival time difference between the pulse leading edge and the trailing edge. (To simplify the problem, only consider the dispersion effect on the nonlinear frequency shifted leading edge and the trailing edge. Also, consider the nonlinear effect only happens from fiber input to \( L_{\text{eff}} \), and dispersion has the effect only from \( L_{\text{eff}} \) to the end of fiber). Which edge of the pulse travels faster?

Solution:

\( L_{\text{eff}} = \frac{1}{\alpha} = 18.87 \text{km}, \) and...
\[
\gamma = \frac{n_2 \omega_0}{c A_{\text{eff}}} = \frac{2m_2}{\lambda A_{\text{eff}}} = \frac{2\pi \times 2.2 \times 10^{-20}}{1550 \times 10^{-9} \times 80 \times 10^{-12}} = 1.1 \times 10^{-3} \text{W}^{-1}
\]

Based on Eq.(2.6.13), \( \Phi_{NL}(t) = \gamma P(t)L_{\text{eff}} = 1.1 \times 10^{-3} \times 18.87 \times 10^3 P(t) = 21P(t) \),

Peak nonlinear phase shift \( \Phi_{NL,\text{peak}} = 21 \times 0.3 = 6.3 \text{rad} \).

Frequency variation can be calculated by
\[
\delta f(t) = \frac{1}{2\pi} \frac{d\Phi_{NL}(t)}{dt}
\]

At pulse leading edge, the derivative of \( d\Phi_{NL}(t) / dt = 6.3 / t_L = 6.3 \times 10^{10} \text{Hz} \)

\[
\delta f(t) = \frac{6.3 \times 10^{10}}{2\pi} \approx 10^{10} \text{Hz} = 10\text{GHz}
\]

At pulse trailing edge \( \delta f(t) = -10\text{GHz} \)

(b) The \( \pm 10\text{GHz} \) optical frequency shift is equivalent to a wavelength shift of \( \pm 0.08\text{nm} \)

(calculated through \( \delta \lambda = -\left(\frac{\lambda^2}{c}\right)\delta f \))

Differential delay can be calculated as
\[
\delta t = D \cdot (L - L_{\text{eff}}) \delta \lambda = 16 \times (80 - 18.87) \times 2 \times 0.08 = 78.34 \text{ps}
\]

The leading edge has a positive frequency shift which is equivalent to a negative wavelength shift (blue shift). Because the dispersion parameter \( D \) is positive (anomalous), group delay is smaller (faster) for shorter wavelength. Thus, the leading edge of the pulse travels faster.
5. When the fiber length is much longer than the effective length \((L >> L_{\text{eff}})\), normalized FWM efficiency \(\eta_{\text{FWM}}\) is only related to fiber loss, dispersion and channel spacing. Consider a degenerate FWM caused by two frequency components in the 1550nm wavelength window separated by 25GHz, and assume the fiber loss is \(\alpha_{\text{dB}} = 0.2\,\text{dB/km}\). Plot FWM efficiency normalized FWM efficiency \(\eta_{\text{FWM}}\) as the function of dispersion parameter \(D\) (from 0 to 20ps/(nm-km)). At which dispersion value, \(\eta_{\text{FWM}}\) is 1%?

Solution,

Based on equations (2.6.24) and (2.6.28),

\[
\Delta \beta_{jkl} = \frac{2\pi D}{\lambda^2}(\lambda_j - \lambda_i)(\lambda_k - \lambda_i), \text{ and } \eta_{\text{FWM}} \approx \frac{\alpha^2}{\Delta \beta_{jkl}^2 + \alpha^2}
\]

For degenerate FWM, \(\lambda_j = \lambda_k\), so that \(\Delta \beta_{jkl} = \frac{2\pi D(\lambda_k - \lambda_i)^2}{\lambda^2} = 2\pi D = 125.5D\)

Where \(\lambda_i = 0.4\,\text{nm}\) was used for 25GHz channel spacing.

\[
\alpha = \alpha_{\text{dB}} / 4.343 = \alpha_{\text{db}} / 4.343 = 0.0461 Np / km = 4.61 \times 10^{-5} Np / m
\]

To match the unit, we also need to convert \(D\) from \([\text{s/(m-m)}]\) to \([\text{ps/(nm-km)}]\), so that

\[
\Delta \beta_{jkl} = \frac{2\pi D(\lambda_k - \lambda_i)^2}{\lambda^2} = 2\pi D = 125.5D \times 10^{-6}
\]

\[
\text{For } \eta_{\text{FWM}} \text{ to be 0.01, } \left(\frac{\Delta \beta_{jkl}}{\alpha}\right)^2 = 99, \text{ that is, } \Delta \beta_{jkl} \approx 10\alpha, \text{ or,}
\]

\[
D = \frac{10\alpha}{125.5 \times 10^{-6}} = \frac{10 \times 4.61 \times 10^{-5}}{125.5 \times 10^{-6}} = 3.67 \left[\frac{\text{ps}}{\text{nm} \cdot \text{km}}\right]
\]