Homework 2 solution:

1. A light beam with $\lambda = 0.63 \mu m$ travels inside a slab optical waveguide by bouncing back and forth between the two interfaces. The refractive index of the waveguide is $n_1 = 1.5$, and the thickness is $d = 3 \mu m$. Assume the lightwave is vertically polarized ($E$ field goes into the paper) with respect to the incidence plane. For the incidence angle $\theta = 60^\circ$, what is the optical phase shift after each roundtrip? What is the evanescent field penetration depth near the glass/air interface?

![Diagram of light beam in a slab optical waveguide]

Solution:

Based on Equation 2.1.8, the optical phase shift of each reflection is

$$\Delta \Phi = -2 \tan^{-1} \left( \frac{n_0 \sqrt{n_1^2 / n_0^2} \sin^2 \theta}{n_1 \cos \theta} \right) = -1.671 \text{ rad.}$$

and the phase delay due to roundtrip propagation is,

$$\Delta \Phi_d = \frac{2\pi}{\lambda \cos \theta} 2dn_1 = \frac{4\pi d \sin \theta}{\lambda \cos \theta} = 179.5 \text{ rad.}$$

Total phase shift is then, $\Delta \Phi = \Delta \Phi_d + \frac{4\pi \lambda}{\lambda \cos \theta} = 179.5 - 1.671 \times 2 = 176 \text{ rad.} \approx 56\pi$

The penetration depth can be found as,

$$z_e = \frac{1}{\alpha} = \frac{\lambda}{2\pi \sqrt{n_1^2 \sin^2 \theta_1 - n_2^2}} = \frac{0.63}{5.21} = 0.121 \mu m$$

2. A step-index optical fiber has the core refractive index $n_1 = 1.50$ and the cladding refractive index $n_2 = 1.497$. The core diameter of the fiber is $d = 8 \mu m$.

(a) What is the numerical aperture of this fiber?
(b) What is the cut-off wavelength $\lambda_c$ of this fiber? (The fiber operates in single mode when the operation wavelength satisfies $\lambda > \lambda_c$)
(c) To carry a blue light at 0.4 $\mu m$ wavelength, approximately how many modes exist in this fiber?

Solution:

(a) $NA = \sqrt{n_1^2 - n_2^2} = \sqrt{1.5^2 - 1.497^2} = 0.0948$

(b) For single mode operation, $V = \frac{2\pi a}{\lambda} \sqrt{n_1^2 - n_2^2} \leq 2.405$, where $a = d/2 = 4 \mu m$.

the cutoff wavelength is, $\lambda_c \geq \frac{2\pi a}{2.405 \sqrt{n_1^2 - n_2^2}} = 0.99 \mu m$
The number of modes is approximately \( M \approx V^2 / 2 = 18 \)

3. For a single-mode step-index fiber, the core index \( n_1 = 1.49 \) and the cladding index \( n_2 = 1.487 \), and the core radius is \( a = 4.5 \, \mu m \).

(a) Please find the wavelength at which \( V = 2 \).

(b) According to Figure 2.3.5, for \( V = 2 \), the normalized propagation constant is approximately \( b \approx 0.4 \). Please find the longitudinal propagation constant of the fundamental mode \( \beta_z \) at this \( b \)-value, and compare this \( \beta_z \) value with free-space \( \beta \)-value corresponding to the refractive index of the core.

(c) Approximately how far away from the core/cladding interface the field amplitude is reduced to 1% compared to its value at core/cladding interface?

Solution:

(a) For \( V = 2 \), the wavelength is,
\[
\lambda = \frac{2\pi a}{\sqrt{n_1^2 - n_2^2}} = \frac{2\pi \times 4 \times 10^{-6}}{0.4 \times 10^{-6}} \approx 0.0948 \, \mu m
\]

(b) By definition, \( b = \left( \frac{\beta_z}{k} \right)^2 - n_2^2 \), where \( k = 2\pi / \lambda \)

\[
\beta_z = \frac{2\pi}{\lambda} \sqrt{b(n_1^2 - n_2^2) + n_2^2} = \frac{2\pi \sqrt{0.4 \times (1.49^2 - 1.487^2) + 1.487^2}}{1.336} = 6.999 \, \mu m^{-1}
\]

The free-space \( \beta \)-value in the core is \( \beta = \frac{2\pi n_1}{\lambda} = \frac{2\pi \times 1.49}{1.336} = 7.0007 \, \mu m^{-1} \), which is slightly higher than \( \beta_z \)-value.

(c) Field attenuation in the cladding is proportional to \( \exp(-W_{in} r) \). Here
\[
W_{in} = \sqrt{\beta_z^2 - \left( \frac{2\pi n_2}{\lambda} \right)^2} = \sqrt{6.999^2 - \left( \frac{2\pi \times 1.487}{1.336} \right)^2} = 0.2815 \, \mu m^{-1}
\]

Thus, for the field amplitude reduction 99%, the distance \( r \) away from the core/cladding interface, is
\[
r = -\frac{\ln(0.01)}{W_{in}} = \frac{\ln(100)}{0.2815} = 16.4 \, \mu m
\]