Homework #3

1. Chromatic dispersion can be represented by a frequency-dependent propagation constant $\beta(\omega)$. Suppose the propagation constant of a single-mode fiber is $\beta(\omega) = A\omega_0 + B(\omega - \omega_0) + C(\omega - \omega_0)^2$, where $\omega_0 = 1.216 \times 10^{15}$ Hz (corresponding to $\lambda = 1550nm$ wavelength).

   (a) If at the optical frequency $\omega_0$, the fiber has a group velocity $v_g = 2 \times 10^8 m/s$ and a chromatic dispersion parameter $D = 15 ps/(nm \cdot km)$, please find the values of $B$ and $C$ in the $\beta(\omega)$ expression.

   (b) Assume an optical pulse is simultaneously carried by two wavelengths separated by 3nm in the 1550nm wavelength window. After 20km transmission through this fiber, approximately what is the pulse separation in the time domain caused by the chromatic dispersion?

Solution:

Since $\beta(\omega) = \beta(\omega_0) + \frac{d\beta}{d\omega}\big|_{\omega=\omega_0} (\omega - \omega_0) + \frac{1}{2} \frac{d^2\beta}{d\omega^2}\big|_{\omega=\omega_0} (\omega - \omega_0)^2$

(a) By definition, the group velocity is $v_g = \frac{1}{\frac{d\beta}{d\omega}}\big|_{\omega=\omega_0} = \frac{1}{B} = 2 \times 10^8 m/s$

Therefore, $B = 5 \times 10^{-9} s/m$

The chromatic dispersion parameter is $\beta_2 = \frac{d^2\beta(\omega)}{d\omega^2}\big|_{\omega=\omega_0} = 2C$

$D = \frac{2\pi c}{\lambda^2} \beta_2 = \frac{4\pi c}{\lambda^2} C$, therefore,

$C = \frac{\lambda^2 D}{4\pi c} = \left(1550 \times 10^{-9}\right)^2 \times 15 \times \frac{10^{-12}}{10^{-9} \times 10^3} = 9.56 \times 10^{-27} s^2/m$

(b) $\Delta t = D \cdot L \cdot \Delta \lambda = 15 \times 20 \times 3 = 900 ps$

2. In a single-mode fiber system the light source has two discrete wavelength components with the same power but separated by 1.2nm. The fiber dispersion parameter in the laser diode wavelength range is $D = 17 ps/(nm \cdot km)$. 
The light source is amplitude modulated by a sinusoid so that the total optical power can be expressed as

\[ P_{opt}(t) = P_0 \left(1 + \cos(2\pi ft)\right) \]  

[note: power carried by each wavelength component is \( P_1(t) = P_2(t) = 0.5P_0(1 + \cos(2\pi ft)) \) at the fiber input]. After transmitting for a certain distance along the fiber, the amplitude modulation on the total optical power may disappear, this is commonly referred to as “carrier fading”.

(a) Explain the physical reason why this carrier fading may occur
(b) If the modulation frequency is \( f = 10GHz \), find the fiber length at which carrier fading happens

Solution:

(a) the two wavelength components propagate in a slightly different speed due to chromatic dispersion. This speed difference is

\[ \delta v_g = D\delta \lambda = 17\text{ ps/ nm/km} \times 1.2\text{ nm} = 20.4\text{ ps/km} \]

When the relative delay is equal to a \( \pi \) phase shift between the two sinusoids, they cancel each other, and thus, no amplitude modulation remains.

(b) If the modulation frequency is \( f = 10GHz \), for \( 2\pi \delta t = \pi \), \( \delta t = 1/(2f) = 5 \times 10^{-11}[s] \),

or, \( \delta v_g L = 5 \times 10^{-11}, \ L = 5 \times 10^{-11} / \delta v_g = \frac{50\text{ ps}}{20.4\text{ ps/km}} = 2.45\text{ km} \),

Thus, carrier fading happens at \( L = 2.45\text{km} \).

(c) The average optical power does not change with modulation frequency, and after 40km, the total fiber loss is \( 50\text{km} \times 0.5dB/km = 25dB \)

Input average optical power is 4mW = 6dBm, so that the output average power is

\[ P_{out} = -19\text{dBm} = 10^{-1.9} = 0.0126\text{mW} \]
3. (a) Polarization mode dispersion of a standard single mode fiber is specified by its mean unit-length differential group delay $DGD_{\text{mean}}$. For $DGD_{\text{mean}} = 0.06 \text{ ps} / \sqrt{\text{km}}$, what is the average differential delay $\Delta \tau_g$ after 1000km of fiber?

(b) A polarization-maintaining (PM) fiber is highly birefringent, and its birefringence is specified by the beat-length. For a PM fiber with 5mm beat length at 1550nm wavelength, what is the differential index $|n_x - n_y|$?

Solution,

(a) Since $\Delta \tau_g = \left( \Delta n_{\text{eff}} / c \right) \sqrt{L} = 0.06 \times \sqrt{1000} = 1.9 \text{ ps}$

(b) Based on Eq. (2.5.30), the relative phase change between the two orthogonal polarization modes is $\Delta \Phi = \left( \omega / c \right) \left| n_x - n_y \right| L = (2\pi / \lambda) \left| n_x - n_y \right| L$. For $\Delta \Phi = 2\pi$, the fiber length is equal to the beat-length: $L = L_p$.

$\left| n_x - n_y \right| = \frac{2\pi}{(2\pi / \lambda) L_p} = \frac{\lambda}{L_p} = \frac{1550 \times 10^{-9}}{5 \times 10^{-3}} = 3.1 \times 10^{-4}$

4. A dispersion-shifted fiber has the effective cross section area $A_{\text{eff}} = 52 \mu m^2$, a nonlinear index $n_2 = 2.2 \times 10^{-20} m^2 / W$, and a loss parameter $\alpha_{\text{dB}} = 0.23 \text{ dB/km}$ at 1550nm wavelength. If the fiber length is 5km and the optical signal is continuous wave (CW), what is the optical power $P$ required at the fiber input to produce nonlinear phase shift of $\Phi_{\text{NL}} = 1 \text{ radians}$? What are the powers required if the fiber length are 100km and 500km (to produce $\Phi_{\text{NL}} = 1 \text{ radians}$)?

Solution:

Based on Eq.(2.6.13), $\Phi_{\text{NL}} = \gamma P(0) \int_0^L e^{-\alpha z} dz = \gamma P(0) \frac{1 - e^{-\alpha L}}{\alpha}$

Here $\gamma = \frac{n_2 c_0}{c A_{\text{eff}}} = \frac{2m_2}{\lambda A_{\text{eff}}} = \frac{2\pi \times 2.2 \times 10^{-20}}{1550 \times 10^{-9} \times 52 \times 10^{-12}} = 1.7 \times 10^{-3} W^{-1}$

$\alpha = 0.23 / 4.343 = 0.053 Np / km = 5.3 \times 10^{-5} Np / m$, and

$\exp(-\alpha L) = \exp(-0.053 \times 5) = 0.7672$

For $\Phi_{\text{NL}} = 1 \text{ radians}$, $P(0) = \frac{\Omega}{\gamma (1 - e^{-\alpha L})} = \frac{5.3 \times 10^{-5}}{1.7 \times 10^{-3} (1 - 0.7672)} = 134 mW$

For $L = 100$km and 500km, $\Phi_{\text{NL}} = \gamma P(0) (1 - e^{-\alpha L}) / \alpha \approx \gamma P(0) / \alpha$

so that $P(0) = \frac{\Phi_{\text{NL}} \alpha}{\gamma} = \frac{5.3 \times 10^{-5}}{1.7 \times 10^{-3}} = 31.2 mW$
A trapezoid-shaped optical pulse shown in the following figure is injected into an optical fiber with the peak power $P_0 = 300mW$, pulse width $T_0 = 1ns$, and equal length of leading edge and trailing edge of 0.1ns ($t_L = t_T = 0.1ns$). A standard single mode fiber is used with $A_{eff} = 80\mu m^2$, a nonlinear index $n_2 = 2.2 \times 10^{-20} m^2 / W$, and a loss parameter $\alpha_{dB} = 0.23dB/km$ at 1550nm wavelength. The fiber length is 80km.

(a) Draw and label the waveforms of nonlinear phase shift, $\Phi_{NL}(t)$, and optical frequency shift $\delta f(t)$ at the fiber output

(b) If the fiber chromatic dispersion is 16ps/(nm-km), please estimate the arrival time difference between the pulse leading edge and the trailing edge. (To simply the problem, only consider the dispersion effect on the nonlinear frequency shifted leading edge and the trailing edge. Also, consider the nonlinear effect only happens from fiber input to $L_{eff}$, and dispersion has the effect only from $L_{eff}$ to the end of fiber). Which edge of the pulse travels faster?

Solution:

$L_{eff} = 1/ \alpha = 18.87 km$, and

$\gamma = \frac{n_2 \omega_0}{c A_{eff}} = \frac{2\pi n_2}{\lambda A_{eff}} = \frac{2\pi \times 2.2 \times 10^{-20}}{1550 \times 10^{-9} \times 80 \times 10^{-12}} = 1.1 \times 10^{-3} W^{-1}$

Based on Eq.(2.6.13), $\Phi_{NL}(t) = \gamma P(t)L_{eff} = 1.1 \times 10^{-3} \times 18.87 \times 10^3 P(t) = 21 P(t)$,

Peak nonlinear phase shift $\Phi_{NL,\text{peak}} = 21 \times 0.3 = 6.3 rad$. 

\[
\begin{align*}
\Phi_{NL}(t) &= \gamma P(t)L_{eff}
\end{align*}
\]
Frequency variation can be calculated by
\[ \delta f(t) = \frac{1}{2\pi} \frac{d\Phi_{NL}(t)}{dt} \]

At pulse leading edge, the derivative of \( d\Phi_{NL}(t) / dt = 6.3 / t_L = 6.3 \times 10^{10} \) Hz

\[ \delta f(t) = \frac{6.3 \times 10^{10}}{2\pi} \approx 10^{10} \text{ Hz} = 10 \text{GHz} \]

At pulse trailing edge \( \delta f(t) = -10 \text{GHz} \)

(b) The ±10GHz optical frequency shift is equivalent to a wavelength shift of ±0.08nm (calculated through \( \delta \lambda = -\left(\frac{\lambda^2}{c}\right)\delta f \))

Differential delay can be calculated as
\[ \delta t = D \cdot (L - L_{eff}) \delta \lambda = 16 \times (80 - 18.87) \times 2 \times 0.08 = 156.5 \text{ ps} \]

The leading edge has a positive frequency shift which is equivalent to a negative wavelength shift (blue shift). Because the dispersion parameter \( D \) is positive (anomalous), group delay is smaller (faster) for shorter wavelength. Thus, the leading edge of the pulse travels faster.