B-Field from Cylindrically Symmetric Current Distributions

Recall we discussed **cylindrically symmetric** charge distributions in Section 4-5. We found that a cylindrically symmetric charge distribution is a function of coordinate ρ **only** (i.e., $\rho_v(\bar{\mathbf{r}}) = \rho_v(\rho)$).

Similarly, we can define a cylindrically symmetric **current** distribution. A current density $\mathbf{J}(\bar{\mathbf{r}})$ is said to be cylindrically symmetric if it points in the direction $\hat{a}_z$ and is a function of coordinate ρ **only**:

$$\mathbf{J}(\bar{\mathbf{r}}) = J_z(\rho)\ \hat{a}_z$$

In other words, $J_\rho = J_\phi = 0$, and J_z is **independent** of both coordinates z and ϕ.

We find that a cylindrically symmetric current density will **always** produce a magnetic flux density of the form:

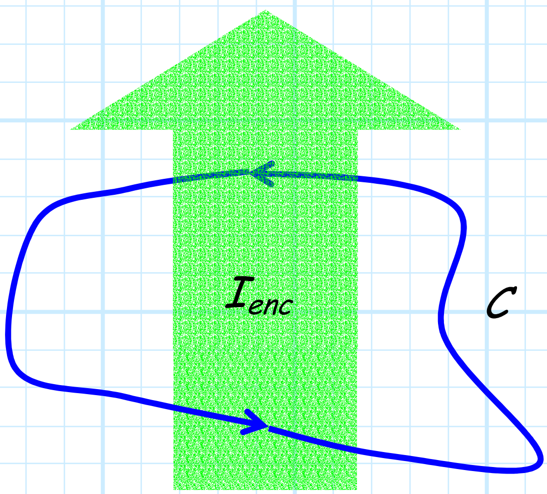
$$\mathbf{B}(\bar{r}) = B_\phi(\rho)\,\hat{a}_\phi$$

In other words, $B_\rho = B_z = 0$, and B_ϕ is independent of **both** coordinates z and ϕ.

Now, lets apply these results to the **integral** form of **Ampere's Law**:

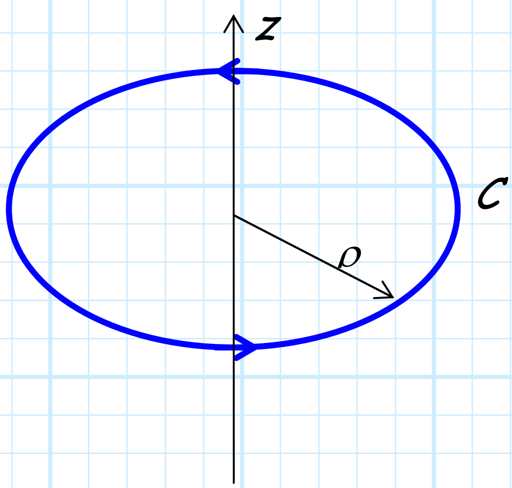
$$\oint_C \mathbf{B}(\bar{\mathbf{r}}) \cdot \overline{d\ell} = \oint_C B_\phi(\rho)\hat{a}_\phi \cdot \overline{d\ell} = \mu_0 I_{enc}$$

where you will recall that I_{enc} is the total **current** flowing **through** the aperture formed by contour C:



Say we choose for contour C a **circle**, centered around the z-axis, with radius ρ.

$$\overline{d\ell} = \hat{a}_\phi \ \rho \, d\phi$$



Amperian Path for Cylindrically Symmetric Current Distributions

This is a special contour, called the **Amperian Path** for **cylindrically symmetric** current densities. To see why it is **special**, let us use it in the cylindrically symmetric form of Ampere's Law:

$$\oint_C B_\phi(\rho)\hat{a}_\phi \cdot \overline{d\ell} = \mu_0 I_{enc}$$

$$\int_0^{2\pi} B_\phi(\rho)\hat{a}_\phi \cdot \hat{a}_\phi \, \rho \, d\phi =$$

$$B_\phi(\rho)\,\rho \int_0^{2\pi} d\phi =$$

$$2\pi\rho \; B_\phi(\rho) = \mu_0 I_{enc}$$

From this result, we can conclude that:

$$B_\phi(\rho) = \frac{\mu_0 \, I_{enc}}{2\pi\rho}$$

Q: But what is I_{enc}?

A: The current flowing **through** the circular aperture formed by contour C!

We of course can determine this by integrating the **current density** $\mathbf{J}(\bar{r})$ across the surface of this circular aperture ($\overline{ds} = \hat{a}_z \; \rho \, d\rho \, d\phi$):

$$
\begin{aligned}
I_{enc} &= \iint_S \mathbf{J}(\bar{\mathbf{r}}) \cdot \overline{ds} \\
&= \int_0^{2\pi} \int_0^{\rho} J_z(\rho')\, \hat{a}_z \cdot \hat{a}_z\, \rho'\, d\rho'\, d\phi \\
&= 2\pi \int_0^{\rho} J_z(\rho')\ \ \rho'\, d\rho'
\end{aligned}
$$

Combining these results, we find that the magnetic flux density $\mathbf{B}(\bar{\mathbf{r}})$ created by a **cylindrically symmetric** current density $\mathbf{J}(\bar{\mathbf{r}})$ is:

$$
\begin{aligned}
\mathbf{B}(\bar{\mathbf{r}}) &= \frac{\mu_0\, I_{enc}}{2\pi\rho}\ \hat{a}_\phi \\
&= \hat{a}_\phi\, \frac{\mu_0}{\rho} \int_0^{\rho} J_z(\rho')\ \ \rho'\, d\rho'
\end{aligned}
$$

For **example**, consider again a **wire** with current I flowing along the z-axis. This is a **cylindrically symmetric** current, and the total current enclosed by an **Amperian path** is clearly I for all ρ (i.e., $I_{enc} = I$).

From the expression above, the magnetic flux density $\mathbf{B}(\bar{\mathbf{r}})$ is therefore:

$$
\mathbf{B}(\bar{\mathbf{r}}) = \frac{\mu_0 I}{2\pi\, \rho}\ \hat{a}_\phi
$$

The **same** result as determined by the **Biot-Savart Law**!