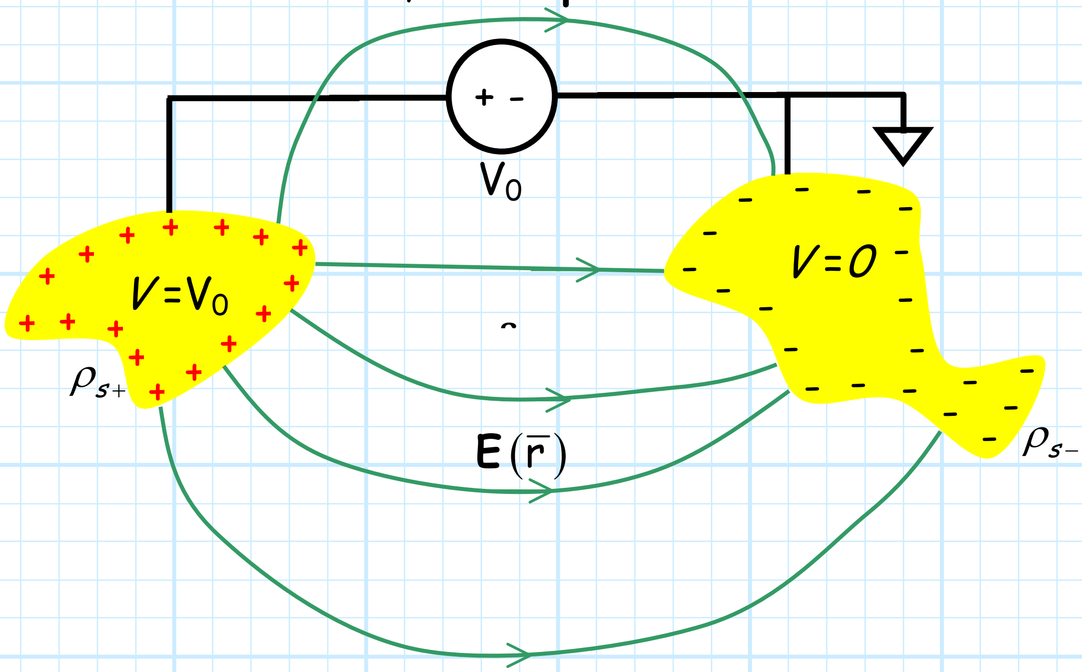
Capacitance

Consider two **conductors**, with a **potential difference** of V volts.



* Since there is a potential difference between the conductors, there must be an **electric potential field** $V(\bar{r})$, and therefore an **electric field** $\mathbf{E}(\bar{r})$ in the region between the conductors.

* Likewise, if there is an electric field, then we can specify an **electric flux density** $\mathbf{D}(\bar{r})$, which we can use to determine the **surface charge density** $\rho_s(\bar{r})$ on each of the conductors.

* We find that if the total net charge on **one** conductor is Q then the charge on the **other** will be equal to –Q.

In other words, the total net charge on each conductor will be **equal** but **opposite**!

Note that this does **not** mean that the surface charge densities on each conductor are equal (i.e., $\rho_{s+}(\bar{r}) \neq \rho_{s-}(\bar{r})$). Rather, it means that:

$$\oiint_{S_+} \rho_{s+}(\bar{r})\, ds = -\oiint_{S_-} \rho_{s-}(\bar{r})\, ds = Q$$

where surface S_+ is the surface surrounding the conductor with the positive charge (and the higher electric potential), while the surface S_- surrounds the conductor with the negative charge.

Q: *How much free* ***charge*** *Q is there on each conductor, and how does this charge relate to the* ***voltage*** *V_0?*

A: We can determine this from the mutual **capacitance** *C* of these conductors!

The mutual **capacitance** between two conductors is **defined** as:

$$C = \frac{Q}{V} \qquad \left[\frac{\text{Coulombs}}{\text{Volt}} \doteq \text{Farad} \right]$$

where Q is the **total charge** on **each conductor**, and *V* is the **potential difference** between each conductor (for our example, $V = V_0$).

Recall that the total charge on a conductor can be determined by **integrating** the surface charge density $\rho_s(\bar{r})$ across the **entire surface** S of a conductor:

$$Q = \oiint_{S_+} \rho_{s+}(\bar{r})\, ds = -\oiint_{S_-} \rho_{s-}(\bar{r})\, ds$$

But recall also that the surface charge density on the surface of a conductor can be determined from the **electric flux density** $\mathbf{D}(\bar{r})$:

$$\rho_s(\bar{r}) = \mathbf{D}(\bar{r}) \cdot \hat{a}_n$$

where $\hat{a}_n$ is a unit vector **normal** to the conductor.

Combining the two equations above, we get:

$$Q = \oiint_{S_+} \mathbf{D}(\bar{r}) \cdot \hat{a}_n ds = -\oiint_{S_-} \mathbf{D}(\bar{r}) \cdot \hat{a}_n ds$$

$$= \oiint_{S_+} \mathbf{D}(\bar{r}) \cdot \overline{ds} = -\oiint_{S_-} \mathbf{D}(\bar{r}) \cdot \overline{ds}$$

where we remember that $\overline{ds} = \hat{a}_n\, ds$.

Hey! This is **no surprise**! We **already** knew that:

$$Q = \oiint_{S} \mathbf{D}(\bar{r}) \cdot \overline{ds}$$

This expression is also know as____________________!!

Note since $\mathbf{D}(\bar{r}) = \varepsilon \mathbf{E}(\bar{r})$ we can also say:

$$Q = \oint\!\!\oint_{S_+} \varepsilon \mathbf{E}(\bar{r}) \cdot \overline{ds}$$

The **potential difference** V between two conductors can likewise be determined as:

$$V = \int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell}$$

where C is **any contour** that leads from one conductor to the other.

Q: Why **any** contour?

A:

We can therefore determine the **capacitance** between two conductors as:

$$C = \frac{Q}{V} = \frac{\oint\!\!\oint_{S_+} \varepsilon \mathbf{E}(\bar{r}) \cdot \overline{ds}}{\int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell}} \qquad [\text{Farad}]$$

Where the contour C must start at **some** point on surface S_+ and end at **some** point on surface S_- .

Note this expression can be written as:

$$Q = C\,V$$

In other words, the charge **stored** by two conductors is equal to the product of their mutual capacitance and the potential difference between them.

Therefore, the **greater** capacitance, the **greater** the amount of **charge** that is stored.

By the way, try taking the **time derivative** of the above equation:

$$\frac{dQ}{dt} = C\,\frac{dV}{dt}$$

$$I = C\,\frac{dV}{dt}$$

Look familiar ?

By the way, the current I in this equation is **displacement current**.