Coordinate Transformations

Say we **know** the location of a point, or the description of some scalar field in terms of **Cartesian** coordinates (e.g., $T(x,y,z)$).

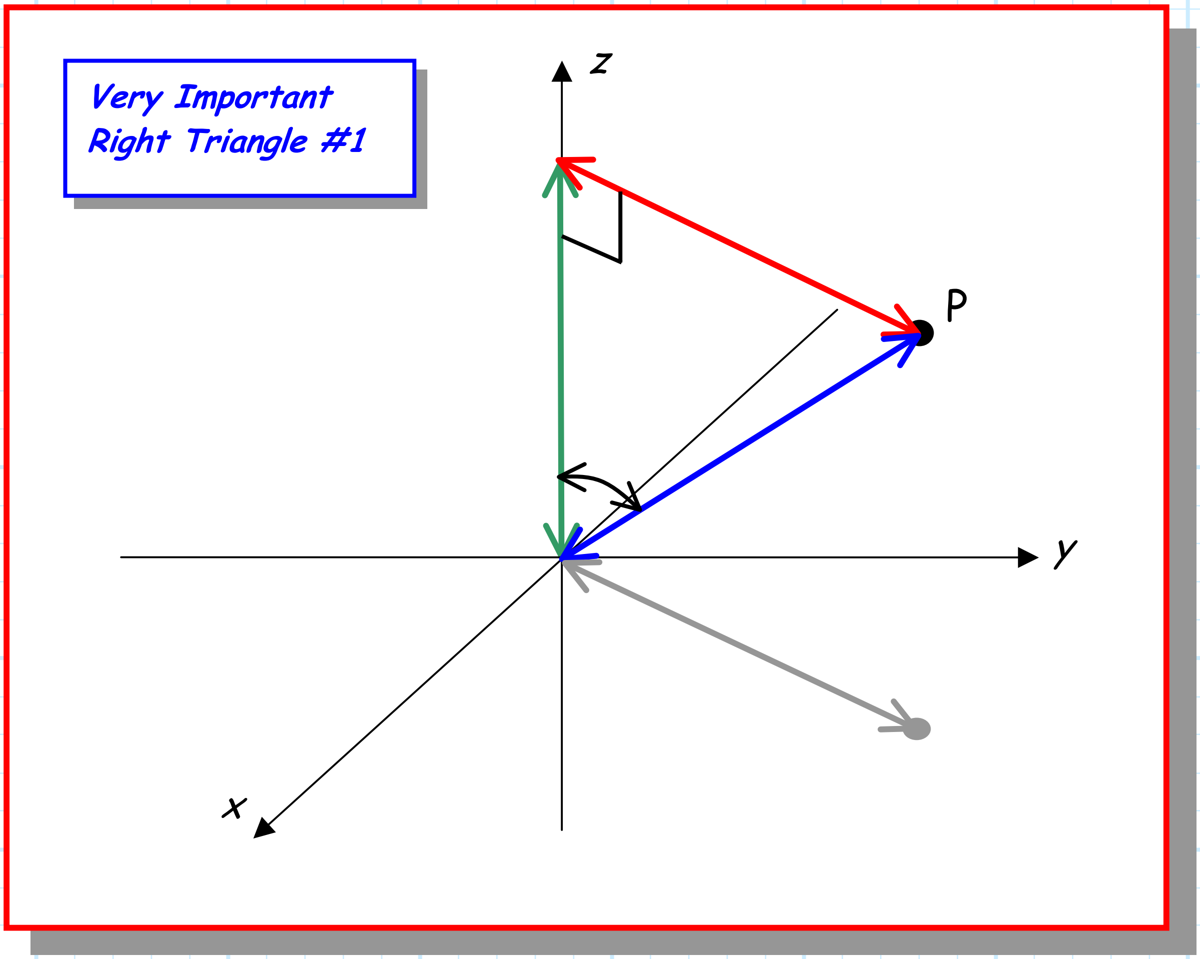
What if we decide to express this point or this scalar field in terms of **cylindrical** or **spherical** coordinates **instead**?

Q: *How do we accomplish this* ***coordinate transformation****?*

A: Easy! We simply apply our knowledge of **trigonometry**.

We see that the coordinate values z, ρ, r, and θ are all variables of a **right triangle**! We can use our knowledge of trigonometry to relate them to each other.

In fact, we can **completely derive** the relationship between **all six** independent coordinate values by considering just **two very important right triangles**! → **Hint:** ***Memorize these 2 triangles!!!***



It is evident from the triangle that, for example:

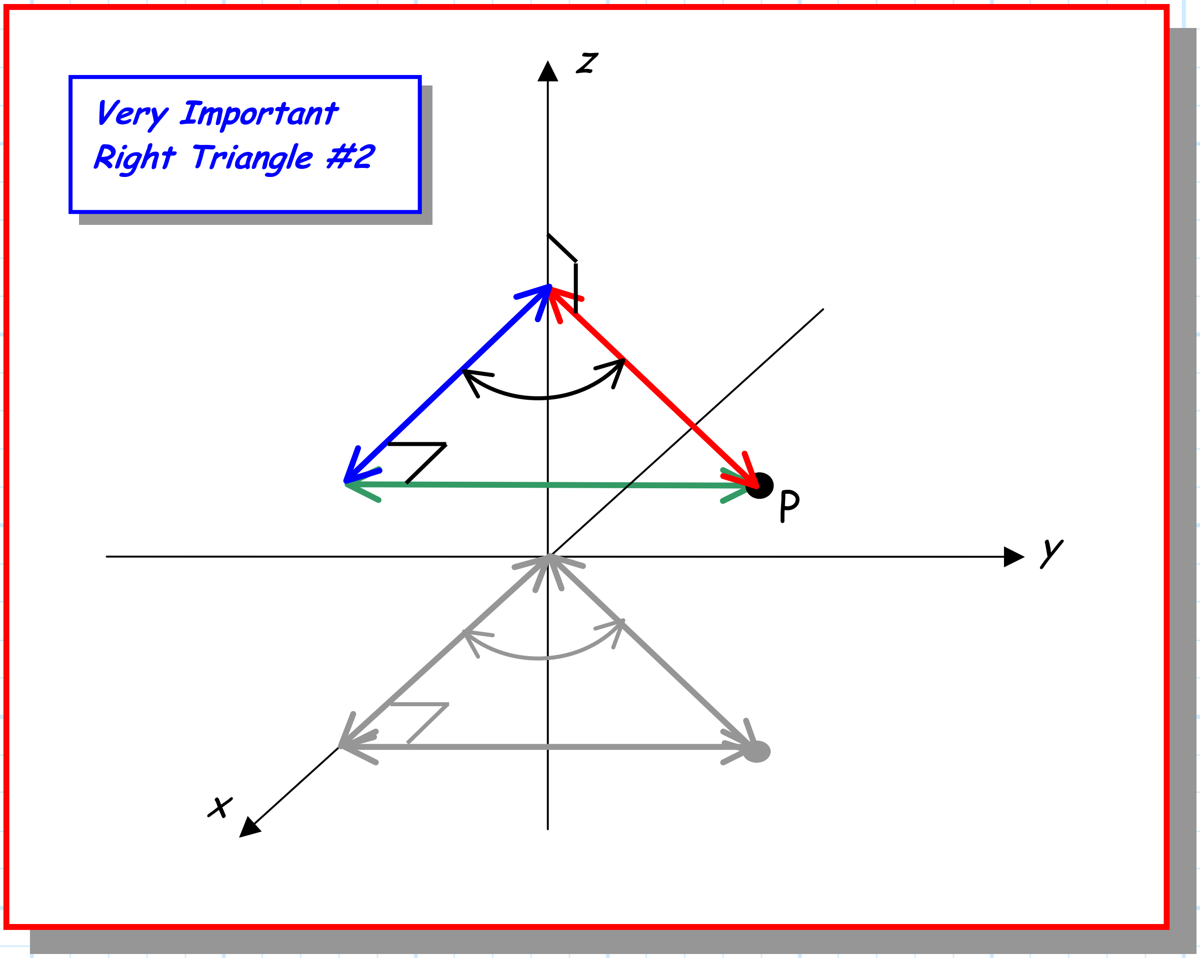
$$z = r\cos\theta = \rho\cot\theta = \sqrt{r^2 - \rho^2}$$

$$\rho = r\sin\theta = z\tan\theta = \sqrt{r^2 - z^2}$$

$$r = \sqrt{\rho^2 + z^2} = \rho\csc\theta = z\sec\theta$$

$$\theta = \tan^{-1}\left[\rho/z\right] = \sin^{-1}\left[\rho/r\right] = \cos^{-1}\left[z/r\right]$$

Likewise, the coordinate values x, y, ρ, and ϕ are **also** related by a **right triangle**!



From the resulting triangle, it is evident that:

$$x = \rho \cos\phi = y \cot\phi = \sqrt{\rho^2 - y^2}$$

$$y = \rho \sin\phi = x \tan\phi = \sqrt{\rho^2 - x^2}$$

$$\rho = \sqrt{x^2 + y^2} = x \sec\phi = y \csc\phi$$

$$\phi = \tan^{-1}\left[y/x \right] = \cos^{-1}\left[x/\rho \right] = \sin^{-1}\left[y/\rho \right]$$

Combining the results of the two triangles allows us to write each coordinate set in terms of each other:

Cartesian and Cylindrical

$$x = \rho \cos\phi \qquad \rho = \sqrt{x^2 + y^2}$$

$$y = \rho \sin\phi \qquad \phi = \tan^{-1}\left[\frac{y}{x}\right] \text{ (be careful !)}$$

$$z = z \qquad z = z$$

Cartesian and Spherical

$$x = r \sin\theta \cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = r \sin\theta \sin\phi \qquad \theta = \cos^{-1}\left[\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right]$$

$$z = r \cos\theta \qquad \phi = \tan^{-1}\left[\frac{y}{x}\right]$$

Cylindrical and Spherical

$$\rho = r \sin\theta$$
$$\phi = \phi$$
$$z = r \cos\theta$$

$$r = \sqrt{\rho^2 + z^2}$$
$$\theta = \tan^{-1}\left[\frac{\rho}{z}\right]$$
$$\phi = \phi$$