Coulomb’s Law for Charge Density

Consider the case where there are multiple point charges present. What is the resulting electrostatic field?

The electric field produced by the charges is simply the vector sum of the electric field produced by each (i.e., superposition!):

\[
E(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}_1'}{|\vec{r} - \vec{r}_1'|^3} + \frac{Q_2}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}_2'}{|\vec{r} - \vec{r}_2'|^3}
\]

Or, more generally, for \( N \) point charges:

\[
E(\vec{r}) = \sum_{n=1}^{N} \frac{Q_n}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}_n'}{|\vec{r} - \vec{r}_n'|^3}
\]
Consider now a volume $V$ that is filled with a "cloud" of charge, descried by **volume charge density** $\rho_v(\mathbf{r})$.

A very small differential volume $d\mathbf{v}$, located at point $\mathbf{r}'$, will thus contain charge $dQ = \rho_v(\mathbf{r}') d\mathbf{v}'$.

This differential charge produces an electric field at point $\mathbf{r}$ equal to:

$$d\mathbf{E}(\mathbf{r}) = \frac{\rho_v(\mathbf{r}') d\mathbf{v}'}{4\pi \varepsilon_0} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

The total electric field at $\mathbf{r}$ (i.e., $\mathbf{E}(\mathbf{r})$) is the summation (i.e., integration) of all the electric field vectors produced by all the little differential charges $dQ$ that make up the charge cloud:

$$\mathbf{E}(\mathbf{r}) = \iiint_V \frac{\rho_v(\mathbf{r}')}{4\pi \varepsilon_0} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d\mathbf{v}'$$

**Note:** The variables of integration are the **primed** coordinates, representing the locations of the charges (i.e., sources).
Similarly, we can show that for **surface** charge:

\[
E(\vec{r}) = \int_S \frac{\rho_s(\vec{r}')}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} d\vec{s}'
\]

And for **line** charge:

\[
E(\vec{r}) = \int_C \frac{\rho_l(\vec{r}')}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} d\ell'
\]