Cylindrical Base Vectors

Cylindrical base vectors are the natural base vectors of a cylinder.

\[ \hat{a}_\rho \] points in the direction of increasing \( \rho \). In other words, \( \hat{a}_\rho \) points away from the \( z \)-axis.

\[ \hat{a}_\phi \] points in the direction of increasing \( \phi \). This is precisely the same base vector we described for spherical base vectors.

\[ \hat{a}_z \] points in the direction of increasing \( z \). This is precisely the same base vector we described for Cartesian base vectors.
It is evident, that like spherical base vectors, the cylindrical base vectors are dependent on position. A vector that points away from the z-axis (e.g., \( \hat{a}_\rho \)), will point in a direction that is dependent on where we are in space!

We can express cylindrical base vectors in terms of Cartesian base vectors. First, we find that:

\[
\begin{align*}
\hat{a}_\rho \cdot \hat{x} &= \cos \phi & \hat{a}_\phi \cdot \hat{x} &= -\sin \phi & \hat{a}_z \cdot \hat{x} &= 0 \\
\hat{a}_\rho \cdot \hat{y} &= \sin \phi & \hat{a}_\phi \cdot \hat{y} &= \cos \phi & \hat{a}_z \cdot \hat{y} &= 0 \\
\hat{a}_\rho \cdot \hat{z} &= 0 & \hat{a}_\phi \cdot \hat{z} &= 0 & \hat{a}_z \cdot \hat{z} &= 1
\end{align*}
\]

We can use these results to write cylindrical base vectors in terms of Cartesian base vectors, or vice versa!

For example,

\[
\hat{a}_\rho = (\hat{a}_\rho \cdot \hat{x}) \hat{x} + (\hat{a}_\rho \cdot \hat{y}) \hat{y} + (\hat{a}_\rho \cdot \hat{z}) \hat{z} = \cos \phi \hat{x} + \sin \phi \hat{y}
\]

or,

\[
\hat{a}_x = (\hat{a}_x \cdot \hat{a}_\rho ) \hat{a}_\rho + (\hat{a}_x \cdot \hat{a}_\phi ) \hat{a}_\phi + (\hat{a}_x \cdot \hat{a}_z ) \hat{a}_z = \cos \phi \hat{a}_\rho - \sin \phi \hat{a}_\phi
\]
Finally, we can write *cylindrical* base vectors in terms of *spherical* base vectors, or vice versa, using the following relationships:

\[
\begin{align*}
\hat{a}_\rho \cdot \hat{a}_r &= \sin \theta \\
\hat{a}_\phi \cdot \hat{a}_r &= 0 \\
\hat{a}_z \cdot \hat{a}_r &= \cos \theta \\
\hat{a}_\rho \cdot \hat{a}_\theta &= \cos \theta \\
\hat{a}_\phi \cdot \hat{a}_\theta &= 0 \\
\hat{a}_z \cdot \hat{a}_\theta &= -\sin \theta \\
\hat{a}_\rho \cdot \hat{a}_\phi &= 0 \\
\hat{a}_\phi \cdot \hat{a}_\phi &= 1 \\
\hat{a}_z \cdot \hat{a}_\phi &= 0 
\end{align*}
\]

e.g.,

\[
\begin{align*}
\hat{a}_r &= (\hat{a}_\rho \cdot \hat{a}_r) \hat{a}_r + (\hat{a}_\phi \cdot \hat{a}_\theta) \hat{a}_\theta + (\hat{a}_z \cdot \hat{a}_\phi) \hat{a}_\phi \\
&= \sin \theta \hat{a}_r + \cos \theta \hat{a}_\theta \\
\hat{a}_\theta &= (\hat{a}_\rho \cdot \hat{a}_r) \hat{a}_r + (\hat{a}_\phi \cdot \hat{a}_\theta) \hat{a}_\theta + (\hat{a}_z \cdot \hat{a}_\phi) \hat{a}_\phi \\
&= \cos \theta \hat{a}_r - \sin \theta \hat{a}_z
\end{align*}
\]