Cylindrically Symmetric Charge Densities

Consider the volume charge densities $\rho_v(\bar{\mathbf{r}})$ that are functions of cylindrical coordinate ρ **only**, e.g.:

$$\rho_v(\bar{\mathbf{r}}) = \frac{1}{\rho^2} \qquad \text{or} \qquad \rho_v(\bar{\mathbf{r}}) = e^{-\rho}$$

We call these types of charge densities **cylindrically symmetric**, as the charge density changes as a function of the distance from the *z*-axis only (i.e., is independent of coordinates ϕ or z).

As a result, the charge distribution in this case looks sort of like a "**fuzzy cylinder**", centered around the *z*-axis!

Using the point form of Gauss's Law, we find the resulting static electric field **must** have the form:

$$\mathbf{E}(\bar{\mathbf{r}}) = E(\rho)\ \hat{a}_\rho \qquad (\text{ for cylindrically symmetric } \rho_v(\bar{\mathbf{r}})\ )$$

Think about what **this** says. It states that the resulting static electric field from a cylindrically symmetric charge density is:

* A function of cylindrical coordinate ρ **only.**

* Points in the direction $\hat{a}_\rho$ (i.e., away from the *z*-axis) at every point.

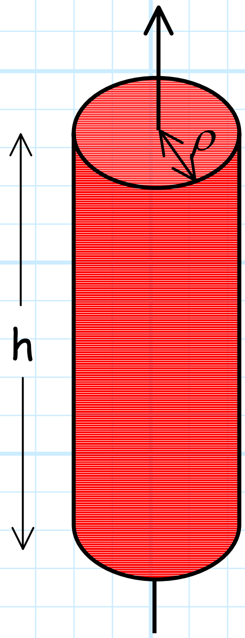
As a result, we can use the **integral form** of Gauss's Law to determine the specific **scalar** function E(ρ) resulting from some **specific**, cylindrically symmetric charge density $\rho_v(\bar{r})$.

Recall the integral form of **Gauss's Law**:

$$\oint\!\!\oint_S \mathbf{E}(\bar{r}) \cdot \overline{ds} = \frac{Q_{enc}}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0} \iiint_V \rho_v(\bar{r})\, dv$$

Say surface S is a cylinder with radius ρ, centered along the *z*-axis. Additionally, this cylinder has a finite length *h*. We call this surface a **Gaussian Surface** for this problem.



We find that, if $\rho_v(\overline{r})$ is **cylindrically symmetric**, then:

$$\oiint_S \mathbf{E}(\overline{r})\cdot\overline{ds} = \int_{-h/2}^{h/2}\int_0^{2\pi} E(\rho)\,\hat{a}_\rho\cdot\hat{a}_\rho\;\rho\,d\phi\,dz \quad \text{side}$$

$$+\int_0^{2\pi}\int_0^{\rho} E(\rho')\,\hat{a}_\rho\cdot\hat{a}_z\;\rho'\,d\rho'\,d\phi \quad \text{top}$$

$$-\int_0^{2\pi}\int_0^{\rho} E(\rho')\,\hat{a}_\rho\cdot\hat{a}_z\;\rho'\,d\rho'\,d\phi \quad \text{bottom}$$

$$= E(\rho)\,\rho\int_{-h/2}^{h/2}\int_0^{2\pi} d\phi\,dz$$

$$= h\,2\pi\rho\;E(\rho)$$

Therefore, from **Gauss's Law**, we get:

$$h\,2\pi\,\rho\,E(\rho) = \frac{Q_{enc}}{\varepsilon_0}$$

Rearranging, we find that the **scalar** function $E(\rho)$ is:

$$E(\rho) = \frac{Q_{enc}}{2\pi\varepsilon_0\,h\rho}$$

The enclosed charge Q_{enc} can be determined for a **cylindrically symmetric** distribution as:

$$
\begin{aligned}
Q_{enc} &= \iiint_V \rho_v(\bar{r})\, dv \\
&= \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^{\rho} \rho_v(\rho')\ \rho'\, d\rho'\, d\phi\, dz \\
&= 2\pi h \int_0^{\rho} \rho_v(\rho')\, \rho'\, d\rho'
\end{aligned}
$$

Therefore, we find that the static electric field produced by a **cylindrically symmetric** charge density is $\mathbf{E}(\bar{r}) = E(\rho)\,\hat{a}_\rho$, where the **scalar** function $E(\rho)$ is:

$$
\begin{aligned}
E(\rho) &= \frac{Q_{enc}}{2\pi\varepsilon_0\, h\rho} \\
&= \frac{1}{\varepsilon_0 \rho} \int_0^{\rho} \rho_v(\rho')\, \rho'\, d\rho'
\end{aligned}
$$

Or, more specifically, we find that the static electric field produced by some **cylindrically symmetric** charge density $\rho_v(\bar{r})$ is:

$$
\begin{aligned}
\mathbf{E}(\bar{r}) &= \frac{Q_{enc}}{2\pi\varepsilon_0\, h\rho}\hat{a}_\rho \\
&= \frac{\hat{a}_\rho}{\varepsilon_0 \rho} \int_0^{\rho} \rho_v(\rho')\, \rho' d\rho'
\end{aligned}
$$

Thus, for a **cylindrically symmetric** charge density, we can find the resulting electric field **without** the difficult integration and evaluation required by **Coulomb's Law**!