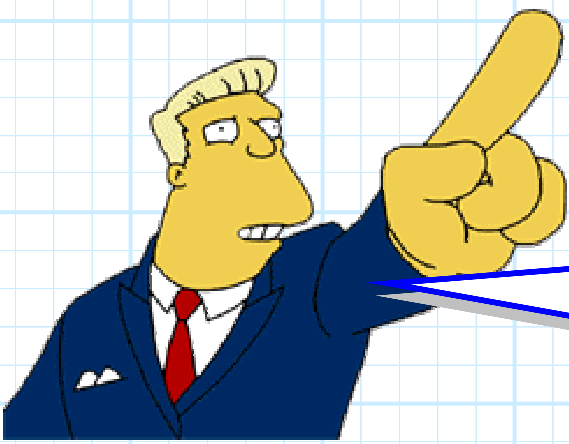
Differential Displacement Vectors

The derivative of a position vector $\bar{r}$, with respect to coordinate value ℓ (where $\ell \in \{x, y, z, \rho, \phi, r, \theta\}$) is expressed as:

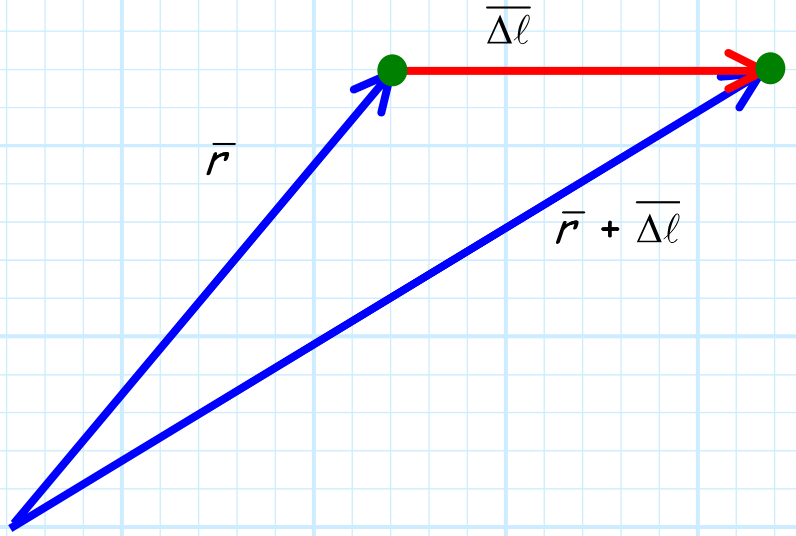
$$
\begin{aligned}
\frac{d\bar{r}}{d\ell} &= \frac{d}{d\ell}\left(x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z\right) \\
&= \frac{d\left(x\,\hat{a}_x\right)}{d\ell} + \frac{d\left(y\,\hat{a}_y\right)}{d\ell} + \frac{d\left(z\,\hat{a}_z\right)}{d\ell} \\
&= \left(\frac{dx}{d\ell}\right)\hat{a}_x + \left(\frac{dy}{d\ell}\right)\hat{a}_y + \left(\frac{dz}{d\ell}\right)\hat{a}_z
\end{aligned}
$$



Q: *Immediately tell me what this incomprehensible result **means** or I shall be forced to pummel you !*

A: The vector above describes the **change** in **position vector** $\bar{r}$ due to a change in coordinate variable ℓ. This change in position vector is itself a vector, with both a **magnitude** and **direction**.

For example, if a point moves such that its coordinate ℓ changes from ℓ to $\ell + \Delta\ell$, then the position vector that describes that point changes from $\bar{r}$ to $\bar{r} + \overline{\Delta\ell}$.



In other words, this small vector $\overline{\Delta\ell}$ is simply a **directed distance** between the point at coordinate ℓ and its new location at coordinate $\ell + \Delta\ell$!

This directed distance $\overline{\Delta\ell}$ is related to the position vector derivative as:

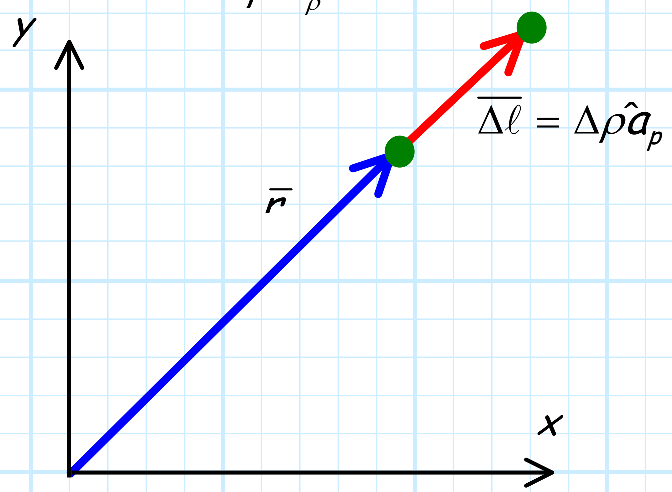
$$\begin{aligned}\overline{\Delta\ell} &= \Delta\ell \; \frac{d\,\bar{r}}{d\ell} \\ &= \Delta\ell\left(\frac{dx}{d\ell}\right)\hat{a}_x + \Delta\ell\left(\frac{dy}{d\ell}\right)\hat{a}_y + \Delta\ell\left(\frac{dz}{d\ell}\right)\hat{a}_z\end{aligned}$$

As an **example**, consider the case when $\ell = \rho$. Since $x = \rho\cos\phi$ and $y = \rho\sin\phi$ we find that:

$$\frac{d\bar{r}}{d\rho} = \frac{dx}{d\rho}\,\hat{a}_x + \frac{dy}{d\rho}\,\hat{a}_y + \frac{dz}{d\rho}\,\hat{a}_z$$
$$= \frac{d(\rho\cos\phi)}{d\rho}\,\hat{a}_x + \frac{d(\rho\sin\phi)}{d\rho}\,\hat{a}_y + \frac{dz}{d\rho}\,\hat{a}_z$$
$$= \cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y$$
$$= \hat{a}_\rho$$

A change in position from coordinates ρ,ϕ,z to $\rho+\Delta\rho,\phi,z$ results in a change in the position vector from $\bar{r}$ to $\bar{r} + \overline{\Delta\ell}$. The vector $\overline{\Delta\ell}$ is a directed distance extending from point ρ,ϕ,z to point $\rho+\Delta\rho,\phi,z$, and is equal to:

$$\overline{\Delta\ell} = \Delta\rho\,\frac{d\bar{r}}{d\rho}$$
$$= \Delta\rho\,\cos\phi\,\hat{a}_x + \Delta\rho\,\sin\phi\,\hat{a}_y$$
$$= \Delta\rho\,\hat{a}_\rho$$



If $\Delta\ell$ is really small (i.e., as it approaches zero) we can define something called a **differential displacement vector** $\overline{d\ell}$:

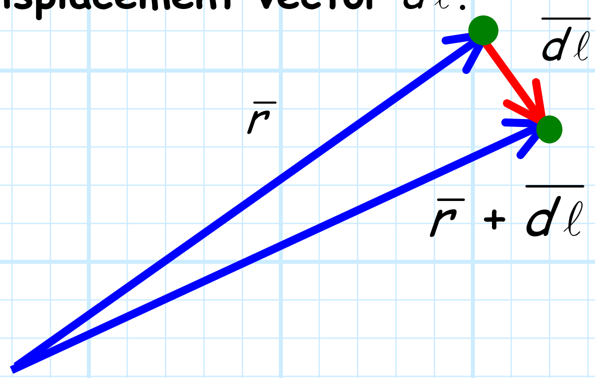
$$\overline{d\ell} \doteq \lim_{\Delta\ell \to 0} \overline{\Delta\ell}$$
$$= \lim_{\Delta\ell \to 0} \left(\frac{d\,\bar{r}}{d\,\ell} \right) \Delta\ell$$
$$= \left(\frac{d\,\bar{r}}{d\,\ell} \right) d\,\ell$$

For example:

$$\overline{d\rho} = \frac{d\bar{r}}{d\rho}\, d\rho = \hat{a}_\rho\, d\rho$$

Essentially, the differential line vector $\overline{d\ell}$ is the **tiny directed distance** formed when a point changes its location by some tiny amount, resulting in a change of one coordinate value ℓ by an equally tiny (i.e., differential) amount $d\ell$.

The **directed distance** between the original location (at coordinate value ℓ) and its new location (at coordinate value $\ell + d\ell$) is the **differential displacement vector** $\overline{d\ell}$.



We will use the differential line vector when evaluating a **line integral**.