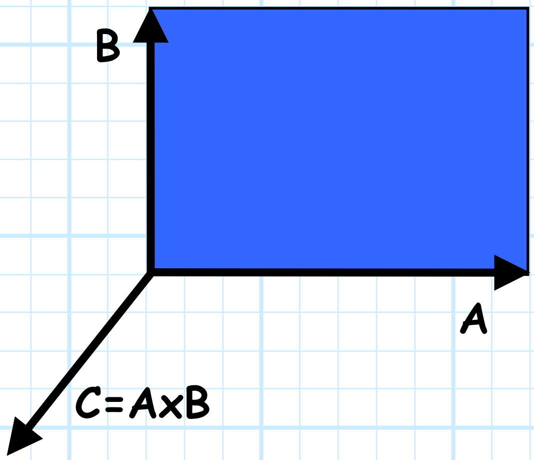
Differential Surface Vectors

Consider a **rectangular surface**, oriented in some arbitrary direction:



We can describe this surface using **vectors**! One vector (say **A**), is a directed distance that denotes the **length** (i.e, magnitude) and **orientation** of one edge of the rectangle, while another directed distance (say **B**) denotes the length and orientation of the other edge.

Say we take the **cross-product** of these two vectors (**AxB=C**).

Q: *What does this vector **C** represent?*

A: Look at the **definition** of cross product!

$$
\begin{aligned}
\mathbf{C} &= \mathbf{A} \times \mathbf{B} \\
&= \hat{a}_n \, |\mathbf{A}| \, |\mathbf{B}| \sin\theta_{AB} \\
&= \hat{a}_n \, |\mathbf{A}| \, |\mathbf{B}|
\end{aligned}
$$

Note that:

$$|\mathbf{C}| = |\mathbf{A}| \, |\mathbf{B}|$$

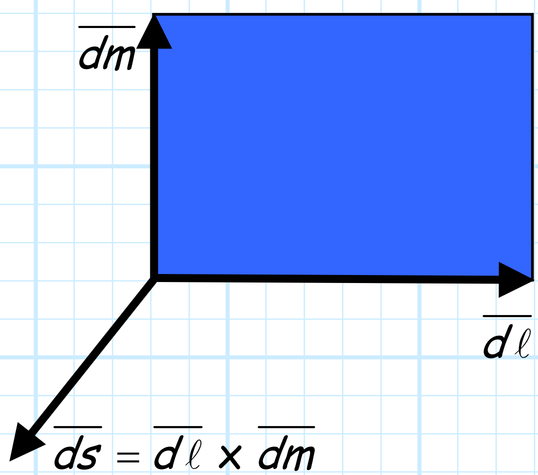
The magnitude of vector $\mathbf{C}$ is therefore product of the lengths of each directed distance—the **area of the rectangle**!

Likewise, $\mathbf{C} \cdot \mathbf{A} = 0$ and $\mathbf{C} \cdot \mathbf{B} = 0$, therefore vector $\mathbf{C}$ is orthogonal (i.e., "**normal**") to the **surface** of the rectangle.

As a result, vector $\mathbf{C}$ indicates **both** the **size** and the **orientation** of the rectangle.

The differential surface vector

For example, consider the **very small** rectangular surface resulting from two differential displacement vectors, say $\overline{d\ell}$ and $\overline{dm}$.



For example, consider the situation if $\overline{d\ell} = \overline{dx}$ and $\overline{dm} = \overline{dy}$:

$$\begin{aligned}\overline{ds} &= \overline{dx} \times \overline{dy} \\ &= \left(\hat{a}_x \times \hat{a}_y \right) dx\, dy \\ &= \hat{a}_z\, dx\, dy\end{aligned}$$

In other words the **differential** surface element has an **area** equal to the product $dx\, dy$, and a **normal vector** that points in the $\hat{a}_z$ direction.

The differential surface vector $\overline{ds}$ specifies the size and orientation of a small (i.e., **differential**) **patch** of area, located on some arbitrary **surface** S.

We will use the differential surface vector in evaluating **surface integrals** of the type:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$