Divergence in Cylindrical and Spherical Coordinate Systems

Consider now the divergence of vector fields when they are expressed in **cylindrical** or **spherical** coordinates:

Cylindrical

$$\nabla \cdot \mathbf{A}(\bar{\mathrm{r}}) = \frac{1}{\rho}\left[\frac{\partial\left(\rho A_\rho(\bar{\mathrm{r}})\right)}{\partial \rho}\right] + \frac{1}{\rho}\frac{\partial A_\phi(\bar{\mathrm{r}})}{\partial \phi} + \frac{\partial A_z(\bar{\mathrm{r}})}{\partial z}$$

Spherical

$$\nabla \cdot \mathbf{A}(\bar{\mathrm{r}}) = \frac{1}{r^2}\left[\frac{\partial\left(r^2 A_r(\bar{\mathrm{r}})\right)}{\partial r}\right] + \frac{1}{r\sin\theta}\left[\frac{\partial\left(\sin\theta\, A_\theta(\bar{\mathrm{r}})\right)}{\partial \theta}\right] + \frac{1}{r\sin\theta}\frac{\partial A_\phi(\bar{\mathrm{r}})}{\partial \phi}$$

Note that, as with the gradient expression, the divergence expressions for cylindrical and spherical coordinate systems are more **complex** than those of Cartesian. Be **careful** when you use these expressions!

For example, consider the vector field:

$$\mathbf{A}(\bar{\mathbf{r}}) = \frac{\sin\theta}{r}\hat{a}_r$$

Therefore, $A_\theta = 0$ and $A_\phi = 0$, leaving:

$$\begin{aligned}\nabla \cdot \mathbf{A}(\bar{\mathbf{r}}) &= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\sin\theta}{r}\right)\right] \\ &= \frac{1}{r^2}\left[\frac{\partial\left(r\sin\theta\right)}{\partial r}\right] \\ &= \frac{1}{r^2}[\sin\theta] = \frac{\sin\theta}{r^2}\end{aligned}$$