Electric Flux Density

Yikes! Things have gotten **complicated**!

In free space, we found that charge $\rho_v(\bar{r})$ creates an electric field $\mathbf{E}(\bar{r})$.

Pretty **simple**! $\rho_v(\bar{r}) \Longrightarrow \mathbf{E}(\bar{r})$

But, if dielectric material is present, we find that charge $\rho_v(\bar{r})$ creates an **initial** electric field $\mathbf{E}_i(\bar{r})$. This electric field in turn **polarizes** the material, forming bound charge $\rho_{vp}(\bar{r})$. This bound charge, however, then creates its **own** electric field $\mathbf{E}_s(\bar{r})$ (sometimes called a **secondary** field), which modifies the initial electric field!

Not so simple! $\rho_v(\bar{r}) \Longrightarrow \mathbf{E}_i(\bar{r}) \Longrightarrow \rho_{vp}(\bar{r}) \Longrightarrow \mathbf{E}_s(\bar{r})$

The **total** electric field created by free charge when dielectric material is present is thus $\mathbf{E}(\bar{r}) = \mathbf{E}_i(\bar{r}) + \mathbf{E}_s(\bar{r})$.

Q: *Isn't there some* ***easier*** *way to account for the effect of dielectric material??*

A: Yes there is! We use the concept of dielectric **permittivity**, and a new vector field called the **electric flux density** $\mathbf{D}(\bar{r})$.

To see how this works, first consider the point form of **Gauss's Law**:

$$\nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \frac{\rho_{vT}(\bar{\mathbf{r}})}{\varepsilon_0}$$

where $\rho_{vT}(\bar{\mathbf{r}})$ is the **total** charge density, consisting of both the **free** charge density $\rho_v(\bar{\mathbf{r}})$ and **bound** charge density $\rho_{vp}(\bar{\mathbf{r}})$:

$$\rho_{vT}(\bar{\mathbf{r}}) = \rho_v(\bar{\mathbf{r}}) + \rho_{vp}(\bar{\mathbf{r}})$$

Therefore, we can write Gauss's Law as:

$$\varepsilon_0 \nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \rho_v(\bar{\mathbf{r}}) + \rho_{vp}(\bar{\mathbf{r}})$$

Recall the **bound** charge density is equal to:

$$\rho_{vp}(\bar{\mathbf{r}}) = -\nabla \cdot \mathbf{P}(\bar{\mathbf{r}})$$

Inserting into the above equation:

$$\varepsilon_0 \nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \rho_v(\bar{\mathbf{r}}) - \nabla \cdot \mathbf{P}(\bar{\mathbf{r}})$$

And rearranging:

$$\varepsilon_0 \nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) + \nabla \cdot \mathbf{P}(\bar{\mathbf{r}}) = \rho_v(\bar{\mathbf{r}})$$

$$\nabla \cdot \left[\varepsilon_0 \mathbf{E}(\bar{\mathbf{r}}) + \mathbf{P}(\bar{\mathbf{r}})\right] = \rho_v(\bar{\mathbf{r}})$$

Note this final result says that the divergence of vector field $\varepsilon_0 \mathbf{E}(\bar{\mathbf{r}}) + \mathbf{P}(\bar{\mathbf{r}})$ is equal to the **free** charge density $\rho_v(\bar{\mathbf{r}})$. Let's define this vector field the **electric flux density** $\mathbf{D}(\bar{\mathbf{r}})$:

$$\text{electric flux density} \quad \mathbf{D}(\bar{\mathbf{r}}) \doteq \varepsilon_0 \mathbf{E}(\bar{\mathbf{r}}) + \mathbf{P}(\bar{\mathbf{r}}) \quad \left[\frac{C}{m^2} \right]$$

Therefore, we can write a **new** form of Gauss's Law:

$$\nabla \cdot \mathbf{D}(\bar{\mathbf{r}}) = \rho_v(\bar{\mathbf{r}})$$

This equation says that the electric flux density $\mathbf{D}(\bar{\mathbf{r}})$ **diverges** from **free** charge $\rho_v(\bar{\mathbf{r}})$. In other words, the source of electric flux density is free charge $\rho_v(\bar{\mathbf{r}})$--and free charge **only**!

* The electric field $\mathbf{E}(\bar{\mathbf{r}})$ is created by **both** free charge and bound charge within the dielectric material.

* However, the electric flux density $\mathbf{D}(\bar{\mathbf{r}})$ is created by **free** charge **only**—the bound charge within the dielectric material makes no difference with regard to $\mathbf{D}(\bar{\mathbf{r}})$!

But wait! We can simplify this further. Recall that the polarization vector is related to electric field by susceptibility $\chi_e(\bar{\mathbf{r}})$:

$$\mathbf{P}(\bar{\mathbf{r}}) = \varepsilon_0 \chi_e(\bar{\mathbf{r}}) \mathbf{E}(\bar{\mathbf{r}})$$

Therefore the electric flux density is:

$$\begin{aligned} \mathbf{D}(\bar{\mathbf{r}}) &= \varepsilon_0 \mathbf{E}(\bar{\mathbf{r}}) + \varepsilon_0 \chi_e(\bar{\mathbf{r}}) \mathbf{E}(\bar{\mathbf{r}}) \\ &= \varepsilon_0 \left(1 + \chi_e(\bar{\mathbf{r}})\right) \mathbf{E}(\bar{\mathbf{r}}) \end{aligned}$$

We can further simplify this by defining the **permittivity** of the medium (the dielectric material):

$$\text{permittivity } \varepsilon(\bar{\mathbf{r}}) \doteq \varepsilon_0 \left(1 + \chi_e(\bar{\mathbf{r}})\right)$$

And can further define **relative** permittivity:

$$\text{relative permittivity } \varepsilon_r(\bar{\mathbf{r}}) \doteq \frac{\varepsilon(\bar{\mathbf{r}})}{\varepsilon_0} = 1 + \chi_e(\bar{\mathbf{r}})$$

Note therefore that $\varepsilon(\bar{\mathbf{r}}) = \varepsilon_r(\bar{\mathbf{r}})\, \varepsilon_0$.

We can thus write a **simple** relationship between electric flux density and electric field:

$$\begin{aligned}\mathbf{D}(\bar{\mathbf{r}}) &= \varepsilon(\bar{\mathbf{r}})\mathbf{E}(\bar{\mathbf{r}}) \\ &= \varepsilon_0\, \varepsilon_r(\bar{\mathbf{r}})\mathbf{E}(\bar{\mathbf{r}})\end{aligned}$$

Like conductivity $\sigma(\bar{\mathbf{r}})$, permittivity $\varepsilon(\bar{\mathbf{r}})$ is a fundamental **material** parameter. Also like conductivity, it relates the electric field to another vector field.

Thus, we have an **alternative** way to view electrostatics:

1. Free charge $\rho_v(\bar{\mathbf{r}})$ creates electric flux density $\mathbf{D}(\bar{\mathbf{r}})$.

2. The electric field can be then determined by simply dividing $\mathbf{D}(\bar{\mathbf{r}})$ by the material permittivity $\varepsilon(\bar{\mathbf{r}})$ (i.e., $\mathbf{E}(\bar{\mathbf{r}}) = \mathbf{D}(\bar{\mathbf{r}})/\varepsilon(\bar{\mathbf{r}})$).

$$\rho_v(\bar{\mathbf{r}}) \Rightarrow \mathbf{D}(\bar{\mathbf{r}}) \Rightarrow \mathbf{E}(\bar{\mathbf{r}})$$