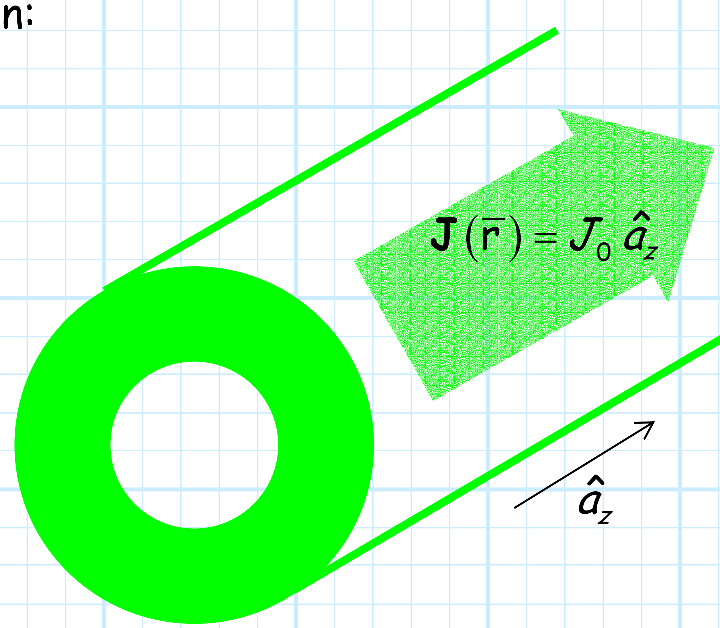
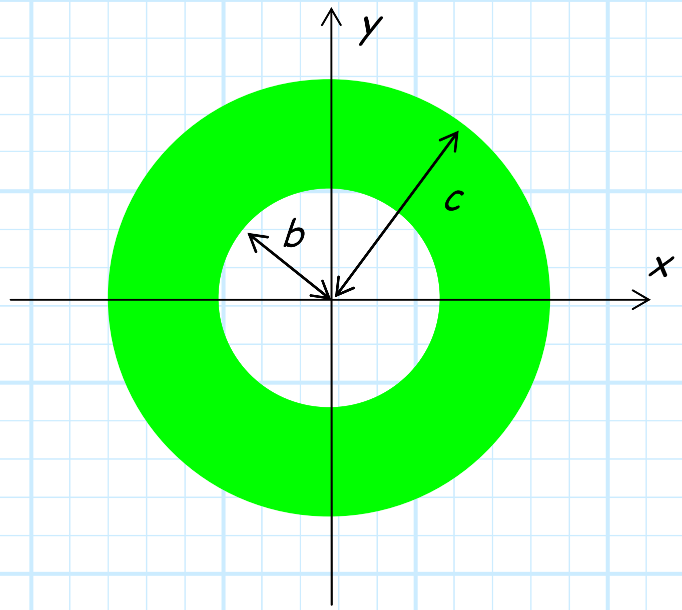
Example: A Hollow Tube of Current

Consider a **hollow cylinder** of uniform current, flowing in the $\hat{a}_z$ direction:





The **inner** surface of the hollow cylinder has radius b, while the **outer** surface has radius c.

The current density in the hollow cylinder is **uniform**, thus we can express current density $\mathbf{J}(\bar{\mathbf{r}})$ as:

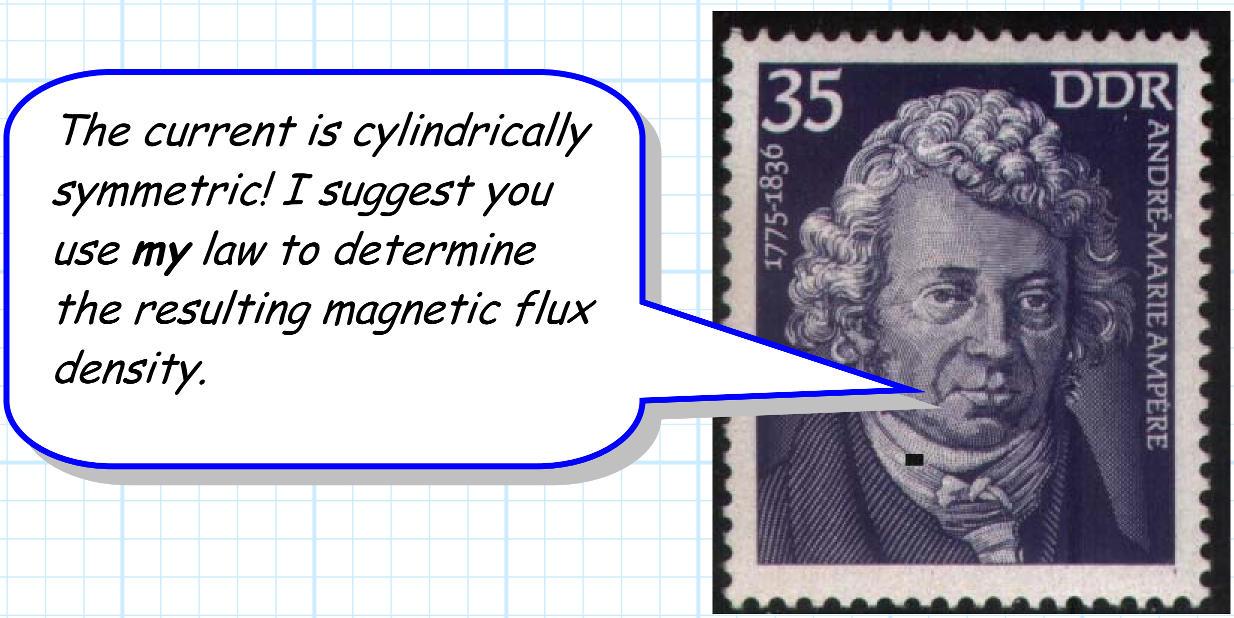
$$\mathbf{J}(\bar{\mathbf{r}}) = \begin{cases} 0 & \rho < b \\ J_0 \hat{a}_z & b < \rho < c \\ 0 & \rho > c \end{cases} \qquad \left[\frac{Amps}{m^2}\right]$$

Q: *What magnetic flux density* $\mathbf{B}(\bar{\mathbf{r}})$ *is produced by this current density* $\mathbf{J}(\bar{\mathbf{r}})$*?*

A: We **could** use the Biot-Savart Law to determine $\mathbf{B}(\bar{\mathbf{r}})$, but note that $\mathbf{J}(\bar{\mathbf{r}})$ is **cylindrically symmetric**!

In other words, current density $\mathbf{J}(\bar{\mathbf{r}})$ has the form:

$$\mathbf{J}(\bar{\mathbf{r}}) = J_z(\rho)\,\hat{a}_z$$



Recall using **Ampere's Law,** we determined that **cylindrically symmetric** current densities produce magnetic flux densities of the form:

$$\mathbf{B}(\bar{r}) = \frac{\mu_0\, I_{enc}}{2\pi\rho}\ \hat{a}_\phi$$

$$= \hat{a}_\phi\ \frac{\mu_0}{\rho} \int_0^{\rho} J_z(\rho')\ \rho'\, d\rho'$$

Therefore, we must evaluate the integral for the current density in this case. Because of the piecewise nature of the current density, we must evaluate the integral for **three** different cases:

1) when the radius of the Amperian path is **less than *b*** (i.e., $\rho < b$).

2) when the radius of the Amperian path is **greater than *b*** but **less than *c*** (i.e., $b < \rho < c$).

3) when the radius of the Amperian path is **greater than *c*.**

$\rho < b$

Note for $\rho < b$, $\mathbf{J}(\bar{r}) = 0$ and therefore the integral is **zero**:

$$\int_0^{\rho} J_z(\rho')\ \rho'\, d\rho' = \int_0^{\rho} 0\ \ \rho'\, d\rho' = 0$$

and therefore:

$$\begin{aligned}\mathbf{B}(\bar{\mathbf{r}}) &= \hat{a}_\phi \frac{\mu_0}{\rho}\, 0 \\ &= 0 \qquad\qquad \text{for} \quad \rho < b\end{aligned}$$

Thus, the magnetic flux density in the hollow region of the cylinder is **zero**!

$b < \rho < c$

Note for $b < \rho < c$, $\mathbf{J}(\bar{\mathbf{r}}) = J_0\, \hat{a}_z$ (i.e., $J_z(\rho) = J_0$) and therefore:

$$\begin{aligned}\int_0^\rho J_z(\rho')\ \rho' d\rho' &= \int_0^b J_z(\rho')\ \rho' d\rho' + \int_b^\rho J_z(\rho')\ \rho' d\rho' \\ &= \int_0^b 0\ \ \rho' d\rho' + \int_b^\rho J_0\ \ \rho' d\rho' \\ &= 0 + J_0 \int_b^\rho \rho' d\rho' \\ &= J_0 \left(\frac{\rho^2}{2} - \frac{b^2}{2}\right) \\ &= J_0 \left(\frac{\rho^2 - b^2}{2}\right)\end{aligned}$$

and therefore the magnetic flux density in the non-hollow portion of the cylinder is:

$$\mathbf{B}(\bar{\mathbf{r}}) = \hat{a}_\phi \frac{\mu_0}{\rho} \; J_0\left(\frac{\rho^2 - b^2}{2}\right) \qquad \text{for} \quad b < \rho < c$$

$\rho > c$

Note that outside the cylinder (i.e., $\rho > c$), the current density $\mathbf{J}(\bar{\mathbf{r}})$ is again **zero**, and therefore:

$$\begin{aligned}\int_0^\rho J_z(\rho')\,\rho'\,d\rho' &= \int_0^b J_z(\rho')\,\rho'\,d\rho' + \int_b^c J_z(\rho')\,\rho'\,d\rho' + \int_c^\rho J_z(\rho')\,\rho'\,d\rho' \\ &= \int_0^b 0 \;\; \rho'\,d\rho' + \int_b^c J_0 \;\; \rho'\,d\rho' + \int_c^\rho 0 \;\; \rho'\,d\rho' \\ &= 0 + J_0 \int_b^c \rho'\,d\rho' + 0 \\ &= J_0\left(\frac{c^2}{2} - \frac{b^2}{2}\right) \\ &= J_0\left(\frac{c^2 - b^2}{2}\right)\end{aligned}$$

Thus, the magnetic flux density **outside** the current cylinder is:

$$\mathbf{B}(\bar{\mathbf{r}}) = \hat{a}_\phi \frac{\mu_0}{\rho} \; J_0\left(\frac{c^2 - b^2}{2}\right) \qquad \text{for} \quad c > \rho$$

Summarizing, we find that the **magnetic flux density** produced by this hollow tube of current is:

$$\mathbf{B}(\bar{\mathrm{r}}) = \begin{cases} 0 & \rho < b \\ \dfrac{J_0\ \mu_0}{\rho}\left(\dfrac{\rho^2 - b^2}{2}\right)\hat{a}_\phi & b < \rho < c \\ \dfrac{J_0\ \mu_0}{\rho}\left(\dfrac{c^2 - b^2}{2}\right)\hat{a}_\phi & \rho > c \end{cases} \qquad \left[\frac{Webers}{m^2}\right]$$

We can find an **alternative** expression by determining the total **current** flowing through this cylinder (let's call this current I_0). We of course can determine I_0 by performing the **surface integral** of the current density $\mathbf{J}(\bar{\mathrm{r}})$ across the cross sectional surface S of the cylinder:

$$\begin{aligned} I_0 &= \iint_S \mathbf{J}(\bar{\mathrm{r}}) \cdot \overline{ds} \\ &= \int_0^{2\pi}\int_b^c J_0\ \hat{a}_z \cdot \hat{a}_z\ \rho\, d\rho\, d\phi \\ &= J_0 \int_0^{2\pi}\int_b^c \rho\, d\rho\, d\phi \\ &= J_0\, \pi\left(c^2 - b^2\right) \end{aligned}$$

Therefore, we can conclude that:

$$J_0 = \frac{I_0}{\pi\left(c^2 - b^2\right)}$$

Inserting this into the expression for the magnetic flux density, we find:

$$\mathbf{B}(\bar{\mathbf{r}}) = \begin{cases} 0 & \rho < b \\ \dfrac{I_0\ \mu_0}{2\pi\rho}\left(\dfrac{\rho^2 - b^2}{c^2 - b^2}\right)\hat{a}_\phi & b < \rho < c \\ \dfrac{I_0\ \mu_0}{2\pi\rho}\ \hat{a}_\phi & \rho > c \end{cases} \qquad \left[\frac{Webers}{m^2}\right]$$

Note the field outside of the cylinder ($\rho > c$) behaves precisely as would the field from a wire of current I_0!