Example: Ampere's Law of Force

Let's again consider Ampere's Law of Force in the following form:

\[
\text{d} \mathbf{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi R^2} \left( \left( \mathbf{d} \mathbf{l}_1 \cdot \hat{\mathbf{a}}_{21} \right) \mathbf{d} \mathbf{l}_2 - \left( \mathbf{d} \mathbf{l}_1 \cdot \mathbf{d} \mathbf{l}_2 \right) \hat{\mathbf{a}}_{21} \right)
\]

\[
= \left( \frac{\mu_0 I_1 I_2}{4\pi R^2} \right) \left( \mathbf{d} \mathbf{l}_1 \cdot \hat{\mathbf{a}}_{21} \right) \mathbf{d} \mathbf{l}_2 - \left( \frac{\mu_0 I_1 I_2}{4\pi R^2} \right) \left( \mathbf{d} \mathbf{l}_1 \cdot \mathbf{d} \mathbf{l}_2 \right) \hat{\mathbf{a}}_{21}
\]

It is apparent that we can consider the force on filament 1 to consist of two forces, i.e.:

\[
\text{d} \mathbf{F}_1 = \text{d} \mathbf{F}_1^a + \text{d} \mathbf{F}_1^b
\]

where

\[
\text{d} \mathbf{F}_1^a = \left( \frac{\mu_0 I_1 I_2}{4\pi R^2} \right) \left( \mathbf{d} \mathbf{l}_1 \cdot \hat{\mathbf{a}}_{21} \right) \mathbf{d} \mathbf{l}_2
\]

and

\[
\text{d} \mathbf{F}_1^b = - \left( \frac{\mu_0 I_1 I_2}{4\pi R^2} \right) \left( \mathbf{d} \mathbf{l}_1 \cdot \mathbf{d} \mathbf{l}_2 \right) \hat{\mathbf{a}}_{21}
\]
Therefore, the force on filament 1 has a component in the direction $\overrightarrow{d\ell_2}$ (i.e., in the direction of current filament 2), and a component in the direction $-\hat{a}_{21}$.

So, let’s consider several examples:

**Example 1:** Filament 2 points toward filament 1

Therefore, since $\overrightarrow{d\ell_2} = |\overrightarrow{d\ell_2}| \hat{a}_{21}$:

\[
(\overrightarrow{d\ell_1} \cdot \hat{a}_{21}) \overrightarrow{d\ell_2} = (\overrightarrow{d\ell_1} \cdot \hat{a}_{21}) |\overrightarrow{d\ell_2}| \hat{a}_{21}
\]

\[
= (\overrightarrow{d\ell_1} \cdot \overrightarrow{d\ell_2}) \hat{a}_{21}
\]

\[
= (\overrightarrow{d\ell_1} \cdot \overrightarrow{d\ell_2}) \hat{a}_{21}
\]

and therefore:
\[ d\mathbf{F}_1^a = - \left( \frac{\mu_0 I_1 I_2}{4\pi R^2} \right) \left( \mathbf{d}\mathbf{l}_1 \cdot \mathbf{\hat{a}}_{21} \right) \mathbf{d}\mathbf{l}_2 \]

\[ = - \left( \frac{\mu_0 I_1 I_2}{4\pi R^2} \right) \left( \mathbf{d}\mathbf{l}_1 \cdot \mathbf{d}\mathbf{l}_2 \right) \mathbf{\hat{a}}_{21} \]

\[ = - d\mathbf{F}_1^b \]

And thus:

\[ d\mathbf{F}_1 = d\mathbf{F}_1^a + d\mathbf{F}_1^b \]

\[ = - d\mathbf{F}_1^b + d\mathbf{F}_1^b = 0 \]

In other words, if filament 2 points at filament 1, then the force on filament 1 is zero, regardless of the orientation of filament 1.

Another way of saying this is that only the component of \( I_2 \mathbf{d}\mathbf{l}_2 \) that is orthogonal to \( \mathbf{\hat{a}}_{21} \) can exert force on filament 1.

**Example 2:** Filament 1 is parallel to filament 2

- Normal component exerts force.
- Tangential component exerts no force.
Therefore, 
\[ \overrightarrow{d\ell_1} \cdot \hat{a}_{21} = 0 \]

so:

\[ d\mathbf{F}_1^a = \left( \frac{\mu_0 I_1 I_2}{4\pi R^2} \right) (\overrightarrow{d\ell_1} \cdot \hat{a}_{21}) \overrightarrow{d\ell_2} = 0 \]

But,

\[ \overrightarrow{d\ell_1} \cdot \overrightarrow{d\ell_2} \neq 0 \]

therefore:

\[ d\mathbf{F}_1^b = -\left( \frac{\mu_0 I_1 I_2}{4\pi R^2} \right) (\overrightarrow{d\ell_1} \cdot \overrightarrow{d\ell_2}) \hat{a}_{21} \neq 0 \]

Thus, \[ d\mathbf{F}_1 = d\mathbf{F}_1^b \], applying a force in the direction \(-\hat{a}_{21}\) !

Filament 1 is attracted to filament 2!

For the same reasons, filament 2 is attracted to filament 1:
But, we find that the two filaments repel if they point in opposite directions:

Example 3: Filament 1 is parallel to $\hat{a}_{21}$ and orthogonal to filament 2.

Therefore,

$$\overrightarrow{d\ell_1} \cdot \overrightarrow{d\ell_2} = 0$$

so:

$$\overrightarrow{dF_1^b} = -\left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overrightarrow{d\ell_1} \cdot \overrightarrow{d\ell_2}\right) \hat{a}_{21} = 0$$

But,

$$\overrightarrow{d\ell_1} \cdot \hat{a}_{21} \neq 0$$

thus:

$$\overrightarrow{dF_1^a} = \left(\frac{\mu_0 I_1 I_2}{4\pi R^2}\right)\left(\overrightarrow{d\ell_1} \cdot \hat{a}_{21}\right) \overrightarrow{d\ell_2} \neq 0$$
Therefore, \( dF_1 = dF_1^a \), applying a force in the direction \( \overrightarrow{dl_2} \):

![Diagram](image)

Note however, the force on filament 2 is zero!

**Example 4:** Filament 1 is orthogonal to \( \hat{a}_{21} \) and orthogonal to filament 2.

In this case, we find:

\[
\overrightarrow{dl_1} \cdot \hat{a}_{21} = 0
\]

so:

\[
dF_1^a = \left( \frac{\mu_0 I_1 I_2}{4\pi R^2} \right) \left( \overrightarrow{dl_1} \cdot \hat{a}_{21} \right) \overrightarrow{dl_2} = 0
\]

Likewise,

\[
\overrightarrow{dl_1} \cdot \overrightarrow{dl_2} = 0
\]

thus:

\[
dF_1^b = - \left( \frac{\mu_0 I_1 I_2}{4\pi R^2} \right) \left( \overrightarrow{dl_1} \cdot \overrightarrow{dl_2} \right) \hat{a}_{21} = 0
\]
Therefore, the total force on filament 1 is zero:

\[ d\mathbf{F}_1 = d\mathbf{F}^a_1 + d\mathbf{F}^b_1 = 0 \]

For the same reasons, we find that the force on filament 2 due to filament 1 is also zero (i.e., \( d\mathbf{F}_2 = 0 \)).