Example: The Uniform, Infinite Line of Current

Consider electric current $I$ flowing along the $z$-axis from $z = -\infty$ to $z = \infty$. What magnetic flux potential $\mathbf{B}(\mathbf{r})$ is created by this current?

We can determine the magnetic flux density by applying the Biot-Savart Law:

$$\mathbf{d\ell} = \hat{z} \, dz'$$

$$\mathbf{\bar{r}} = x \, \hat{a}_x + y \, \hat{a}_y + z \, \hat{a}_z$$

$$= \rho \cos \phi \, \hat{a}_x + \rho \sin \phi \, \hat{a}_y + z \, \hat{a}_z$$

$$\mathbf{\bar{r}'} = z' \, \hat{a}_z \quad (x' = 0, y' = 0)$$

$$|\mathbf{\bar{r}} - \mathbf{\bar{r}'}| = \sqrt{\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi + (z - z')^2}$$

$$= \sqrt{\rho^2 + (z - z')^2}$$

We can determine the magnetic flux density by applying the Biot-Savart Law:
\[ \mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_\mathcal{C} \frac{\mathbf{d}l' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]

\[ = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \hat{a}_z \times \left[ \rho \cos \phi \hat{a}_x + \rho \sin \phi \hat{a}_y + (z - z') \hat{a}_z \right] \frac{dz'}{\left[ \rho^2 + (z - z')^2 \right]^{3/2}} \]

\[ = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho \cos \phi \hat{a}_y - \rho \sin \phi \hat{a}_x}{\left[ \rho^2 + (z - z')^2 \right]^{3/2}} \, dz' \]

\[ = \frac{\mu_0 I}{4\pi} \left( \rho \cos \phi \hat{a}_y - \rho \sin \phi \hat{a}_x \right) \int_{-\infty}^{\infty} \frac{du}{\left[ \rho^2 + u^2 \right]^{3/2}} \]

\[ = \frac{\mu_0 I}{4\pi} \left( \rho \hat{a}_\phi \right) \bigg|_{-\infty}^{\infty} \frac{u}{\rho^2 \sqrt{\rho^2 + u^2}} \]

\[ = \frac{\mu_0 I}{4\pi} \left( \rho \hat{a}_\phi \right) \frac{2}{\rho^2} \]

\[ = \frac{\mu_0 I}{2\pi \rho} \hat{a}_\phi \]

Therefore, the magnetic flux density created by a "wire" with current \( I \) flowing along the \( z \)-axis is:

\[ \mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi \rho} \hat{a}_\phi \]
Think about what this expression tells us about magnetic flux density:

* The magnitude of \( \mathbf{B}(\mathbf{r}) \) is proportional to \( 1/\rho \), therefore magnetic flux density diminishes as we move farther from "wire".

* The direction of \( \mathbf{B}(\mathbf{r}) \) is \( \hat{a}_\phi \). In other words, the magnetic flux density points in the direction around the wire.

Plot of vector field \( \mathbf{B}(\mathbf{r}) \) on the \( x-y \) plane, resulting from current \( I \) flowing along the \( z \)-axis

\( \bigcirc \) = current \( I \) flowing out of this page.
Or, plotting in 3-D: