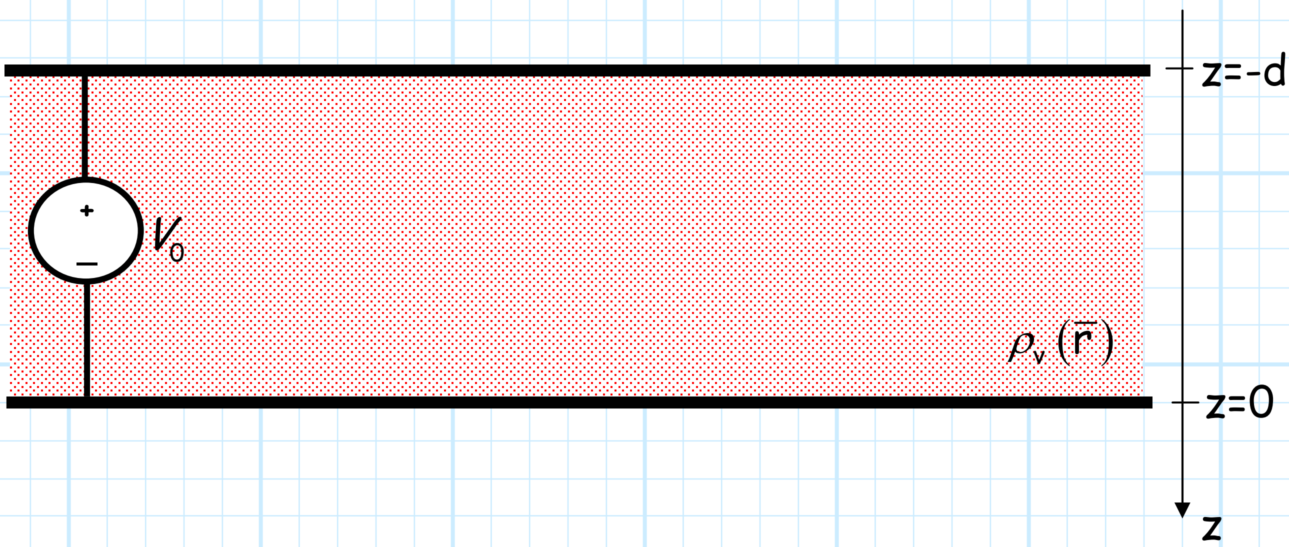
Example: Charge Filled Parallel Plates

Consider now a problem similar to the previous example (i.e., dielectric filled parallel plates), with the exception that the space between the infinite, conducting parallel plates is filled with **free charge**, with a density:

$$\rho_v(\bar{r}) = -z\,\varepsilon_0 \qquad (-d < z < 0)$$



Q: *How do we determine the fields within the parallel plates for **this** problem?*

A: Same as before! However, since the charge density between the plates is **not** equal to zero, we recognize that the electric potential field must satisfy **Poisson's equation**:

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon_0}$$

For the specific charge density $\rho_v(\bar{r}) = -z\,\varepsilon_0$:

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon_0} = z$$

Since **both** the charge density and the plate geometry are **independent** of coordinates x and y, we know the electric potential field will be a function of coordinate z **only** (i.e., $V(\bar{r}) = V(z)$).

Therefore, Poisson's equation becomes:

$$\nabla^2 V(z) = \frac{\partial^2 V(z)}{\partial z^2} = z$$

We can solve this differential equation by first **integrating** both sides:

$$\int \frac{\partial^2 V(z)}{\partial z^2} dz = \int z\,dz$$

$$\frac{\partial V(z)}{\partial z} = \frac{z^2}{2} + C_1$$

And then integrating a **second time**:

$$\int \frac{\partial V(\bar{r})}{\partial z} dz = \int \left(\frac{z^2}{2} + C_1 \right) dz$$

$$V(\bar{r}) = \frac{z^3}{6} + C_1 z + C_2$$

Note that this expression for $V(\bar{r})$ **satisfies** Poisson's equation for this case. The question remains, however: what are the values of **constants** C_1 and C_2?

We find them in the same manner as before—**boundary conditions**!

Note the boundary conditions for **this** problem are:

$$V(z=-d)=V_0$$

$$V(z=0)=0$$

Therefore, we can construct **two** equations with **two** unknowns:

$$V(z=-d)=V_0=\frac{(-d)^3}{6}+C_1(-d)+C_2$$

$$V(z=0)=0=\frac{(0)^3}{6}+C_1(0)+C_2$$

It is evident that $C_2 = 0$, therefore constant C_1 is:

$$C_1=-\left(\frac{V_0}{d}+\frac{d^2}{6}\right)$$

The **electric potential field** between the two plates is therefore:

$$V(\bar{r}) = \frac{z^3}{6} - \left(\frac{V_0}{d} + \frac{d^2}{6}\right) z \qquad (-d < z < 0)$$

Performing our **sanity check**, we find:

$$\begin{aligned} V(z=\text{-d}) &= \frac{(-\text{d})^3}{6} - \left(\frac{V_0}{d} + \frac{d^2}{6}\right)(\text{-d}) \\ &= \frac{-\text{d}^3}{6} + V_0 + \frac{d^3}{6} \\ &= V_0 \ \checkmark \end{aligned}$$

and

$$\begin{aligned} V(z=0) &= \frac{(0)^3}{6} - \left(\frac{V_0}{d} + \frac{d^2}{6}\right)(0) \\ &= 0 + 0 + 0 \\ &= 0 \ \checkmark \end{aligned}$$

From this result, we can determine the **electric field** $\mathbf{E}(\bar{r})$, the **electric flux density** $\mathbf{D}(\bar{r})$, and the **surface charge density** $\rho_s(\bar{r})$, as before.

Note, however, that the permittivity of the material between the plates is ε_0, as the "dielectric" between the plates is **free-space**.