Example: Coordinate Transformations

Say we have denoted a **point** in space (using **Cartesian** Coordinates) as $P(x=-3,\ y=-3,\ z=2)$.

Let's **instead** define this **same** point using **cylindrical** coordinates ρ, ϕ, z:

$$\rho = \sqrt{x^2+y^2} = \sqrt{(-3)^2+(-3)^2} = 3\sqrt{2}$$

$$\phi = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left[\frac{-3}{-3}\right] = \tan^{-1}[1] = 45^\circ$$

$$z = 2$$

Therefore, the location of this point can **perhaps** be defined **also** as $P(\rho = 3\sqrt{2},\ \phi = 45^\circ,\ z = 2)$.

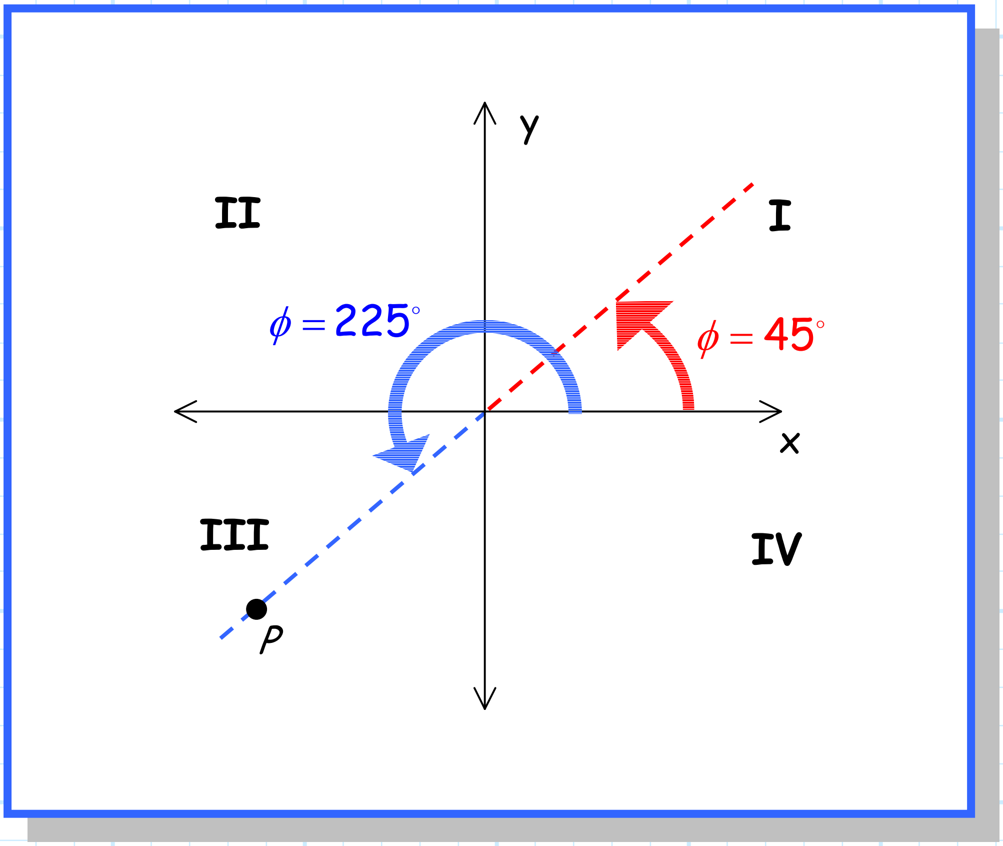
Q: ***Wait!*** *Something has gone* ***horribly wrong****. Coordinate* $\phi = 45^\circ$ *indicates that point P is located in* ***quadrant I****, whereas the coordinates x =-3, y =-3 tell us it is in fact in* ***quadrant III****!*

A: The problem is our interpretation of the **inverse tangent!**

Remember that $0 \leq \phi < 360^\circ$, so that we must do a **four quadrant** inverse tangent. Your calculator likely only does a **two quadrant** inverse tangent (i.e., $90 \leq \phi \leq -90^\circ$), so **be careful**!

Therefore, if we **correctly** find the coordinate ϕ:

$$\phi = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left[\frac{-3}{-3}\right] = 225^\circ$$



The location of point P can be expressed as **either** $P(x=-3,\ y=-3,\ z=2)$ or $P(\rho = 3\sqrt{2},\ \phi = 225^\circ,\ z = 2)$.

We can also perform a **coordinate transformation** on a **scalar field**. For **example**, consider the scalar field (i.e., scalar function):

$$g(\rho,\phi,z) = \rho^3 \sin\phi\, z$$

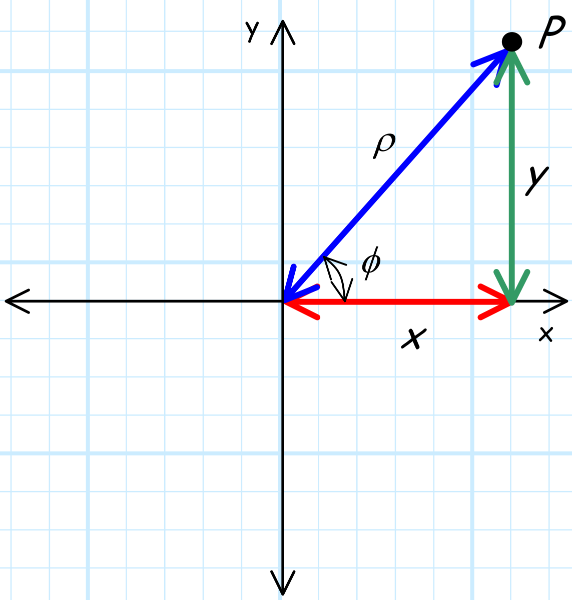
Lets try to **rewrite** this function in terms of **Cartesian** coordinates. We first note that since $\rho = \sqrt{x^2 + y^2}$,

$$\rho^3 = \left(x^2 + y^2\right)^{3/2}$$

Now, what about **sinϕ**? We know that $\phi = \tan^{-1}[y/x]$, thus we might be tempted to write:

$$\sin\phi = \sin\left[\tan^{-1}\left[\frac{y}{x}\right]\right]$$

Although **technically** correct, this is one **ugly** expression. We can instead turn to one of the **very important right triangles** that we discussed earlier:



From **this** triangle, it is apparent that:

$$\sin\phi =$$

As a result, the scalar field can be written in **Cartesian** coordinates as:

$$g(x,y,z) = \left(x^2+y^2\right)^{3/2} \frac{y}{\sqrt{x^2+y^2}}\, z$$
$$= \left(x^2+y^2\right) yz$$

Remember, although the scalar fields:

$$g(x,y,z) = \left(x^2+y^2\right) yz$$

and:

$$g(\rho,\phi,z) = \rho^3 \sin\phi\, z$$

look very different, they are in fact **exactly** the same functions—only expressed using different **coordinate variables**.

For **example**, if you **evaluate** each of the scalar fields at the **point** described earlier in the handout, you will get **exactly the same** result!

$$g(x=-3, y=-3, z=2) = -108$$

$$g(\rho=3\sqrt{2}, \phi=225^\circ, z=2) = -108$$