Example: Dielectric Filled Parallel Plates

Consider two infinite, parallel conducting plates, spaced a distance \( d \) apart. The region between the plates is filled with a dielectric \( \varepsilon \). Say a voltage \( V_0 \) is placed across these plates.

Q: **What electric potential field \( V(\vec{r}) \), electric field \( E(\vec{r}) \) and charge density \( \rho_s(\vec{r}) \) is produced by this situation?**

A: We must solve a boundary value problem! We must find solutions that:

- a) Satisfy the differential equations of electrostatics (e.g., Poisson’s, Gauss’s).
- b) Satisfy the electrostatic boundary conditions.
Q: Yikes! Where do we even start?

A: We might start with the electric potential field \( V(r) \), since it is a scalar field.

a) The electric potential function must satisfy Poisson’s equation:

\[
\nabla^2 V(r) = \frac{-\rho_v(r)}{\varepsilon}
\]

b) It must also satisfy the boundary conditions:

\[
V(z = -d) = V_0 \quad V(z = 0) = 0
\]

Consider first the dielectric region \((-d < z < 0)\). Since the region is a dielectric, there is no free charge, and:

\[
\rho_v(r) = 0
\]

Therefore, Poisson’s equation reduces to Laplace’s equation:

\[
\nabla^2 V(r) = 0
\]

This problem is greatly simplified, as it is evident that the solution \( V(r) \) is independent of coordinates \( x \) and \( y \). In other words, the electric potential field will be a function of coordinate \( z \) only:

\[
V(r) = V(z)
\]
This makes the problem much easier! Laplace’s equation becomes:

\[ \nabla^2 V (\mathbf{r}) = 0 \]
\[ \nabla^2 V (z) = 0 \]
\[ \frac{\partial^2 V (z)}{\partial z^2} = 0 \]

Integrating both sides of Laplace’s equation, we get:

\[ \int \frac{\partial^2 V (z)}{\partial z^2} \, dz = \int 0 \, dz \]
\[ \frac{\partial V (z)}{\partial z} = C_1 \]

And integrating again we find:

\[ \int \frac{\partial V (z)}{\partial z} \, dz = \int C_1 \, dz \]
\[ V (z) = C_1 z + C_2 \]

We find that the equation \( V (z) = C_1 z + C_2 \) will satisfy Laplace’s equation (try it!). We must now apply the boundary conditions to determine the value of constants \( C_1 \) and \( C_2 \).

We know that the value of the electrostatic potential at every point on the top \( (z = -d) \) plate is \( V (-d) = V_0 \), while the electric potential on the bottom plate \( (z = 0) \) is zero \( (V (0) = 0) \). Therefore:
\[ V(z = -d) = -C_1d + C_2 = V_0 \]

\[ V(z = 0) = C_1(0) + C_2 = 0 \]

Two equations and two unknowns \((C_1\text{ and } C_2)!\)

Solving for \(C_1\) and \(C_2\) we get:

\[ C_2 = 0 \quad \text{and} \quad C_1 = -\frac{V_0}{d} \]

and therefore, the electric potential field within the dielectric is found to be:

\[ V(\vec{r}) = \frac{-V_0 z}{d} \quad (-d \leq z \leq 0) \]

Before we proceed, let’s do a sanity check!

In other words, let’s evaluate our answer at \(z = 0\) and \(z = -d\), to make sure our result is correct:

\[ V(z = -d) = \frac{-V_0(-d)}{d} = V_0 \quad \checkmark \]

and

\[ V(z = 0) = \frac{-V_0(0)}{d} = 0 \quad \checkmark \]
Now, we can find the electric field within the dielectric by taking the gradient of our result:

$$E(r) = -\nabla V(r) = \frac{V_0}{d} \hat{a}_z \quad (-d \leq z \leq 0)$$

And thus we can easily determine the electric flux density by multiplying by the dielectric of the material:

$$D(r) = \varepsilon E(r) = \varepsilon \frac{V_0}{d} \hat{a}_z \quad (-d \leq z \leq 0)$$

Finally, we need to determine the charge density that actually created these fields!

**Q:** Charge density ?! I thought that we already determined that the charge density $\rho_v(r)$ is equal to zero?

**A:** We know that the free charge density within the dielectric is zero—but there must be charge somewhere, otherwise there would be no fields!
Recall that we found that at a conductor/dielectric interface, the surface charge density on the conductor is related to the electric flux density in the dielectric as:

\[ D_n = \hat{a}_n \cdot \mathbf{D}(\mathbf{r}) = \rho_s(\mathbf{r}) \]

First, we find that the electric flux density on the bottom surface of the top conductor (i.e., at \( z = -d' \)) is:

\[ \mathbf{D}(\mathbf{r}) \bigg|_{z=-d'} = \frac{\varepsilon V_0}{d} \hat{a}_z \bigg|_{z=-d'} = \frac{\varepsilon V_0}{d} \hat{a}_z \]

For every point on bottom surface of the top conductor, we find that the unit vector normal to the conductor is:

\[ \hat{a}_n = \hat{a}_z \]

Therefore, we find that the surface charge density on the bottom surface of the top conductor is:

\[ \rho_{s+}(\mathbf{r}) = \hat{a}_n \cdot \mathbf{D}(\mathbf{r}) \bigg|_{z=-d'} = \hat{a}_z \cdot \hat{a}_z \frac{\varepsilon V_0}{d} = \frac{\varepsilon V_0}{d} \quad (z = -d) \]
Likewise, we find the unit vector \textbf{normal} to the \textit{top} surface of the \textbf{bottom} conductor is (do you see why):

\[ \hat{a}_n = -\hat{a}_z \]

Therefore, evaluating the \textbf{electric flux density} on the top surface of the bottom conductor (i.e., \( z = 0 \)), we find:

\[
\rho_s - (\vec{r}) = \hat{a}_n \cdot \mathbf{D}(\vec{r})\bigg|_{z=0} \\
= -\hat{a}_z \cdot \hat{a}_z \frac{\varepsilon V_0}{d} \\
= \frac{-\varepsilon V_0}{d} \quad (z = 0)
\]

We should \textbf{note} several things about these solutions:

1) \( \nabla \times \mathbf{E}(\vec{r}) = 0 \)

2) \( \nabla \cdot \mathbf{D}(\vec{r}) = 0 \) and \( \nabla^2 V(\vec{r}) = 0 \)

3) \( \mathbf{D}(\vec{r}) \) and \( \mathbf{E}(\vec{r}) \) are \textbf{normal} to the surface of the conductor (i.e., their \textbf{tangential} components are equal to \textbf{zero}).

4) The \textbf{electric field} is precisely the \textbf{same} as that given by using superposition and eq. 4.20 in section 4-5!
I.E.:

\[
E(\vec{r}) = \frac{\rho_{s+}}{2\varepsilon} \hat{a}_z - \frac{\rho_{s-}}{2\varepsilon} \hat{a}_z = \frac{V_0}{d} \hat{a}_z \quad (-d < z < 0)
\]

In other words, the fields \( E(\vec{r}) \), \( D(\vec{r}) \), and \( V(\vec{r}) \) are attributable to charge densities \( \rho_{s+}(\vec{r}) \) and \( \rho_{s-}(\vec{r}) \).