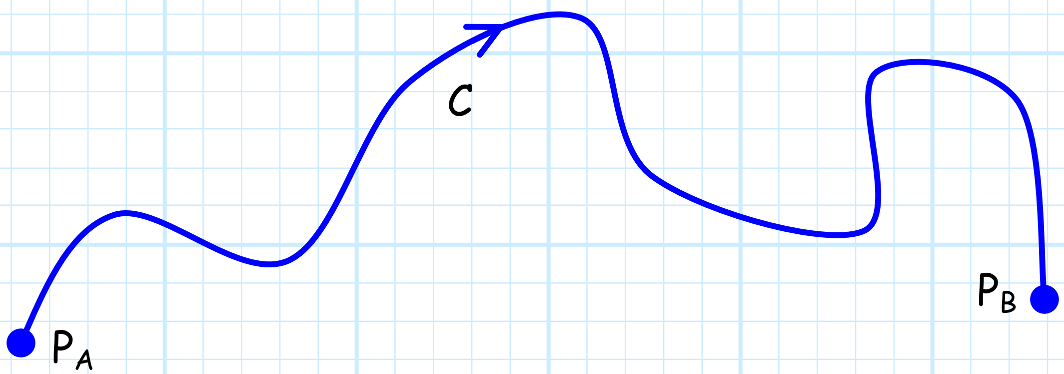
Example: Line Integrals of Conservative Fields

Consider the vector field $\mathbf{A}(\bar{r}) = \nabla\left(x^2 + y^2\right)z$.

Evaluate the contour integral:

$$\int_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell}$$

where $\mathbf{A}(\bar{r}) = \nabla\left(x^2 + y^2\right)z$, and contour C is:



The **beginning** of contour C is the point denoted as:

$$\bar{r}_A = 3\,\hat{a}_x - \hat{a}_y + 4\,\hat{a}_z$$

while the **end** point is denote with position vector:

$$\bar{r}_B = -3\,\hat{a}_x - 2\,\hat{a}_z$$

Note that ordinarily, this would be an **impossible** problem for **us** to do!

But, we note that vector field $\mathbf{A}(\bar{r})$ is **conservative,** therefore:

$$\int_C A(\bar{r})\cdot\overline{d\ell} = \int_C \nabla g(\bar{r})\cdot\overline{d\ell}$$
$$= g(\bar{r}=\bar{r}_B) - g(\bar{r}=\bar{r}_A)$$

For this problem, it is evident that:

$$g(\bar{r}) = \left(x^2+y^2\right)z$$

Therefore, $g(\bar{r}=\bar{r}_A)$ is the **scalar** field evaluated at $x = 3$, $y = -1$, $z = 4$; while $g(\bar{r}=\bar{r}_B)$ is the **scalar** field evaluated at at $x = -3$, $y = 0$, $z = -2$.

$$g(\bar{r}=\bar{r}_A) = \left((3)^2+(-1)^2\right)4 = 40$$

$$g(\bar{r}=\bar{r}_B) = \left((-3)^2+(0)^2\right)(-2) = -18$$

Therefore:

$$\int_C A(\bar{r})\cdot\overline{d\ell} = \int_C \nabla g(\bar{r})\cdot\overline{d\ell}$$
$$= g(\bar{r}=\bar{r}_B) - g(\bar{r}=\bar{r}_A)$$
$$= -18 - 40$$
$$= -58$$