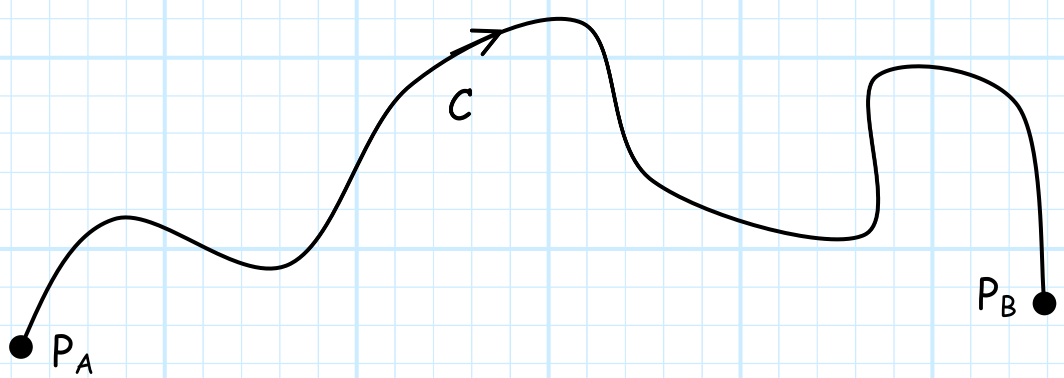
Example: Line Integrals of Conservative Fields

Consider the vector field $\mathbf{A}(\bar{r}) = \nabla\left(x^2 + y^2\right)z$.

Evaluate the line integral:

$$\int_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell}$$

where $\mathbf{A}(\bar{r}) = \nabla\left(x^2 + y^2\right)z$, and contour C is:



The beginning of contour C is the point denoted as:

$$\bar{r}_A = 3\,\hat{a}_x - \hat{a}_y + 4\,\hat{a}_z$$

while the end point is denote with position vector:

$$\bar{r}_B = -3\,\hat{a}_x - 2\,\hat{a}_z$$

Note that ordinarily, this would be an impossible problem for **us** to do!

But, we note that vector field $\mathbf{A}(\bar{r})$ is **conservative,** therefore:

$$\begin{aligned}\int_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell} &= \int_C \nabla g(\bar{\mathrm{r}}) \cdot \overline{d\ell} \\ &= g(\bar{\mathrm{r}}_B) - g(\bar{\mathrm{r}}_A)\end{aligned}$$

For this problem, it is evident that:

$$g(\bar{r}) = \left(x^2 + y^2\right) z$$

Therefore, $g(\bar{r}_A)$ is the scalar field evaluated at x = 3, y = -1, z = 4; while $g(\bar{r}_B)$ is the scalar field evaluated at at x = -3, y = 0, z = -2.

$$g(\bar{r}_A) = \left((3)^2 + (-1)^2\right) 4 = 40$$

$$g(\bar{r}_B) = \left((-3)^2 + (0)^2\right)(-2) = -18$$

Therefore:

$$\begin{aligned}\int_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell} &= \int_C \nabla g(\bar{\mathrm{r}}) \cdot \overline{d\ell} \\ &= g(\bar{\mathrm{r}}_B) - g(\bar{\mathrm{r}}_A) \\ &= -18 - 40 \\ &= -58\end{aligned}$$