Example: Magnetization Currents

Problem:

Consider an infinite cylinder made of magnetic material. This cylinder is centered along the z-axis, has a radius of 2 m, and a permeability of $4\mu_0$.

Inside the cylinder there exists a magnetic flux density:

$$B(\vec{r}) = \frac{8\mu_0}{\rho} \hat{\phi} \quad (\rho \leq 1)$$

Determine the magnetization current $J_{sm}(\vec{r}_s)$ flowing on the surface of this cylinder, as well as the magnetization current $J_m(\vec{r})$ flowing within the volume of this cylinder.

Solution:

First, we note that we must know the magnetization vector $\mathbf{M}(\vec{r})$ in order to find the magnetization currents:

$$J_m(\vec{r}) = \nabla \times \mathbf{M}(\vec{r}) \quad \left[\frac{A}{m^2}\right]$$

$$J_{sm}(\vec{r}_s) = \mathbf{M}(\vec{r}_s) \times \hat{n} \quad \left[\frac{A}{m}\right]$$
But, we must know the **magnetic susceptibility** $\chi_m$ and the magnetic field $H(\vec{r})$ to determine magnetization vector.

$$\mathbf{M}(\vec{r}) = \chi_m \mathbf{H}(\vec{r})$$

Likewise, we need to know the **relative permeability** $\mu_r$ to determine magnetic susceptibility:

$$\chi_m = \mu_r - 1$$

and we need to know the **magnetic flux density** $\mathbf{B}(\vec{r})$ to determine the magnetic field:

$$\mathbf{H}(\vec{r}) = \frac{\mathbf{B}(\vec{r})}{\mu}$$

But guess what! We know the relative permeability $\mu_r$ of the material, as well as the magnetic flux density within it!

$$\mu = 4\mu_0, \quad \therefore \mu_r = 4$$

$$\mathbf{B}(\vec{r}) = \frac{8\mu_0}{\rho} \hat{a}_\phi \quad (\rho \leq 1)$$

Therefore, the **magnetic field** is:

$$\mathbf{H}(\vec{r}) = \frac{\mathbf{B}(\vec{r})}{\mu} = \frac{1}{4\mu_0} \frac{8\mu_0}{\rho} \hat{a}_\phi = \frac{2}{\rho} \hat{a}_\phi$$
and the magnetic susceptibility is:

\[ \chi_m = \mu_r - 1 = 4 - 1 = 3 \]

So the magnetization vector is:

\[ \mathbf{M}(\mathbf{r}) = \chi_m \mathbf{H}(\mathbf{r}) = (3) \frac{2}{\rho} \hat{a}_\phi = \frac{6}{\rho} \hat{a}_\phi \]

Now (finally!) we can determine the magnetization currents:

\[ \mathbf{J}_m(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r}) = \nabla \times \left( \frac{6}{\rho} \hat{a}_\phi \right) = 0 \]

The volume magnetization current density is zero—there is no magnetization current flowing within the cylinder!

\[ \textbf{Q: No magnetization currents!} \]
\[ \text{So we're done right? This problem is solved?} \]
A: Not hardly! Although there are no magnetization currents flowing within the cylinder, there might be magnetization currents flowing on the cylinder surface (i.e., \( J_{sm}(\vec{r}_s) \))!

\[
J_{sm}(\vec{r}_s) = M(\vec{r}_s) \times \hat{a}_n
\]

Note for this problem, the unit vector normal to the surface of the cylinder is \( \hat{a}_n = \hat{a}_\rho \).

Likewise, the magnetization vector evaluated at the cylinder surface (i.e., at \( \rho = 2 \)) is:

\[
M(\vec{r}_s) = M(\rho = 2) = \frac{6}{\rho} \hat{a}_\phi \bigg|_{\rho=2} = 3 \hat{a}_\phi
\]

Therefore, the magnetization current density on the cylinder surface is:

\[
J_{sm}(\rho = 2) = M(\rho = 2) \times \hat{a}_n
= 3 \hat{a}_\phi \times \hat{a}_\rho
= -3 \hat{a}_z \quad [A/m]
\]

Now, we're finally done.