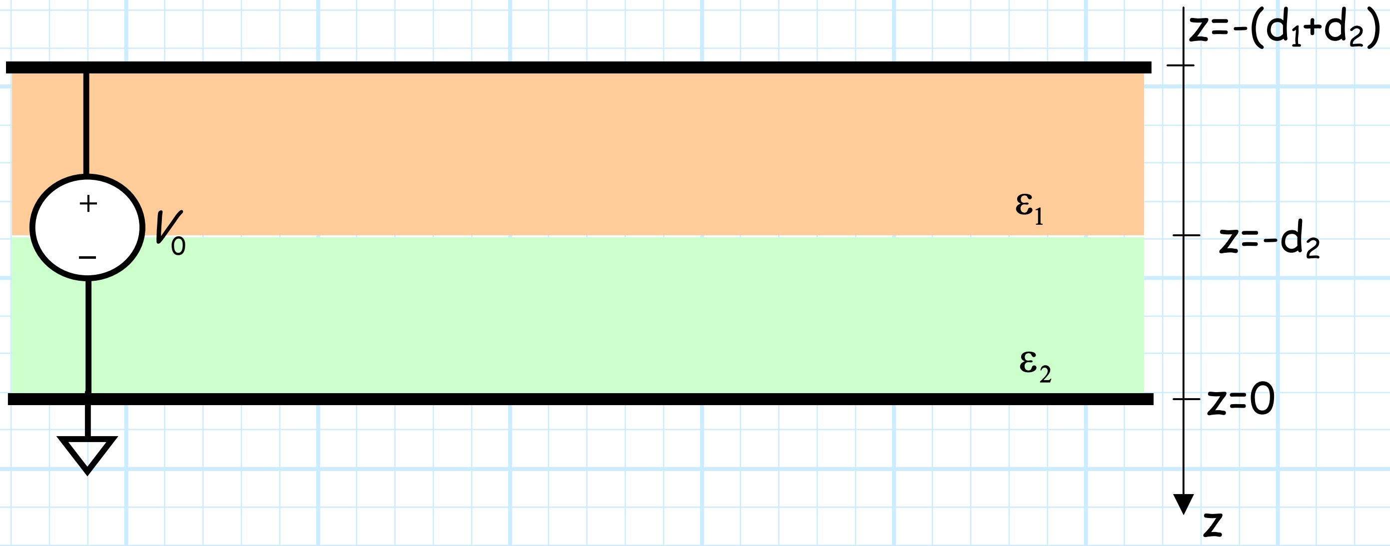
Example: Solutions Involving Inhomogeneous Dielectrics

We again consider the parallel plate problem, only this time the plates are filled with two **different** layers of dielectric !



The **top** dielectric layer has thickness d_1 and permittivity ε_1, while the **bottom** layer has thickness d_2 and permittivity ε_2.

Q: *How do we determine the fields within these dielectric **layers**?*

A: Begin by defining the electric potential function in **each** dielectric:

$$V_1(\bar{r}) \doteq \text{electric potential function in material } \varepsilon_1$$

$$V_2(\bar{r}) \doteq \text{electric potential function in material } \varepsilon_2$$

Note as before, these functions will be **independent** of coordinates x and y. Therefore:

$$V_1(\bar{r}) = V_1(z) \quad \text{and} \quad V_2(\bar{r}) = V_2(z)$$

Each of these functions must satisfy **Laplace's equation**, therefore:

$$\nabla^2 V_1(\bar{r}) = \frac{\partial^2 V_1(z)}{\partial z^2} = 0$$

$$\nabla^2 V_2(\bar{r}) = \frac{\partial^2 V_2(z)}{\partial z^2} = 0$$

We **know** from an earlier handout that these solutions will have the form:

$$V_1(z) = C_{1a}\, z + C_{1b}$$

$$V_2(z) = C_{2a}\, z + C_{2b}$$

YIKES! We now have **FOUR** unknowns (C_{1a}, C_{1b}, C_{2a}, C_{2b})! Therefore, we need **four boundary conditions** to determine these constants.

We know for starters, that :

$$V_1\left(z=-\left(d_1+d_2\right)\right)=V_0=-C_{1a}\left(d_1+d_2\right)+C_{1b}$$

$$V_2\left(z=0\right)=0=C_{2a}\left(0\right)+C_{2b}$$

Therefore, it is evident that C_{2b} = 0. But, what about the other **three** constants?

We need **two more** boundary conditions!

Another boundary condition can be found at the **interface** between the two dielectrics (i.e., at $z=-d_2$). The electric pontential at this boundary must be **the same** for **both** expressions:

$$V\left(z=-d_2\right)=V_1\left(z=-d_2\right)=V_2\left(z=-d_2\right)$$

As a result, we get the boundary equation:

$$V_1\left(z=-d_2\right)=V_2\left(z=-d_2\right)$$
$$C_{1a}\left(-d_2\right)+C_{1b}=C_{2a}\left(-d_2\right)$$

where we have used the fact that C_{2b} = 0.

But, we still need **one more** boundary condition.

Recall a **boundary condition** for dielectric interfaces is:

$$\mathbf{D}_{1n}\left(\bar{\mathbf{r}}\right)=\mathbf{D}_{2n}\left(\bar{\mathbf{r}}\right)$$

We can apply **this** boundary condition as well! Recall:

$$\mathbf{D}(\bar{\mathbf{r}}) = \varepsilon\, \mathbf{E}(\bar{\mathbf{r}}) = -\varepsilon \nabla V(\bar{\mathbf{r}})$$

where for this case:

$$\nabla V_1(\bar{\mathbf{r}}) = \frac{\partial V_1(z)}{\partial z}\hat{a}_z = C_{1a}\,\hat{a}_z$$

and

$$\nabla V_2(\bar{\mathbf{r}}) = \frac{\partial V_2(z)}{\partial z}\hat{a}_z = C_{2a}\,\hat{a}_z$$

Therefore:

$$\mathbf{D}_1(\bar{\mathbf{r}}) = -\varepsilon_1 C_{1a}\,\hat{a}_z$$

$$\mathbf{D}_2(\bar{\mathbf{r}}) = -\varepsilon_2 C_{2a}\,\hat{a}_z$$

Since $\hat{a}_n = \hat{a}_z$, we find from this boundary condition that:

$$D_{1n}(\bar{\mathbf{r}}) = D_{2n}(\bar{\mathbf{r}})$$

$$-\varepsilon_1 C_{1a} = -\varepsilon_2 C_{2a}$$

Now we can **solve** for our constants ! Recall our **four** equations are:

1) $V_0 = -C_{1a}(d_1 + d_2) + C_{1b}$ from $V_1(z = -(d_1 + d_2)) = V_0$

2) $C_{2b} = 0$ from $V_2(z = 0) = 0$

3) $-C_{1a} d_2 + C_{1b} = -C_{2a} d_2$ from $V_1(z = -d_2) = V_2(z = -d_2)$

4) $\varepsilon_1 C_{1a} = \varepsilon_2 C_{2a}$ from $D_{1n}(\bar{r}) = D_{2n}(\bar{r})$

Solving these equations, we find (in addition to C_{2b}=0):

$$C_{1a} = \frac{-V_0 \varepsilon_2}{\varepsilon_1 d_2 + \varepsilon_2 d_1}$$

and

$$C_{1b} = \frac{V_0 (\varepsilon_1 - \varepsilon_2) d_2}{\varepsilon_1 d_2 + \varepsilon_2 d_1}$$

and

$$C_{2a} = \frac{-V_0 \varepsilon_1}{\varepsilon_1 d_2 + \varepsilon_2 d_1}$$

Inserting these results into the equations:

$$V_1(z) = C_{1a}\, z + C_{1b}$$

$$V_2(z) = C_{2a}\, z + C_{2b}$$

provides the electric potential field in each region within the parallel plates. From this result, we can determine the **electric field** and the **electric flux density** within each region, as well as the charge density on each plate.