**Example: Solutions Involving Inhomogeneous Dielectrics**

We again consider the parallel plate problem, only this time the plates are filled with two different layers of dielectric!

![Diagram of parallel plates with dielectric layers](image)

The **top** dielectric layer has thickness $d_1$ and permittivity $\varepsilon_1$, while the **bottom** layer has thickness $d_2$ and permittivity $\varepsilon_2$.

**Q:** How do we determine the fields within these dielectric layers?

**A:** Begin by defining the electric potential function in each dielectric:
$V_1(\vec{r})$ = electric potential function in material $\varepsilon_1$

$V_2(\vec{r})$ = electric potential function in material $\varepsilon_2$

Note as before, these functions will be independent of coordinates x and y. Therefore:

$$V_1(\vec{r}) = V_1(z) \quad \text{and} \quad V_2(\vec{r}) = V_2(z)$$

Each of these functions must satisfy Laplace's equation, therefore:

$$\nabla^2 V_1(\vec{r}) = \frac{\partial^2 V_1(z)}{\partial z^2} = 0$$

$$\nabla^2 V_2(\vec{r}) = \frac{\partial^2 V_2(z)}{\partial z^2} = 0$$

We know from an earlier handout that these solutions will have the form:

$$V_1(z) = C_{1a} z + C_{1b}$$

$$V_2(z) = C_{2a} z + C_{2b}$$

YIKES! We now have FOUR unknowns ($C_{1a}$, $C_{1b}$, $C_{2a}$, $C_{2b}$)!

Therefore, we need four boundary conditions to determine these constants.

We know for starters, that:
\[ V_1(z = -(d_1 + d_2)) = V_0 = -C_{1a}(d_1 + d_2) + C_{1b} \]

\[ V_2(z = 0) = 0 = C_{2a}(0) + C_{2b} \]

Therefore, it is evident that \( C_{2b} = 0 \). But, what about the other three constants?

We need two more boundary conditions!

Another boundary condition can be found at the interface between the two dielectrics (i.e., at \( z = -d_2 \)). The electric potential at this boundary must be the same for both expressions:

\[ V(z = -d_2) = V_1(z = -d_2) = V_2(z = -d_2) \]

As a result, we get the boundary equation:

\[ V_1(z = -d_2) = V_2(z = -d_2) \]

\[ C_{1a}(-d_2) + C_{1b} = C_{2a}(-d_2) \]

where we have used the fact that \( C_{2b} = 0 \).

But, we still need one more boundary condition.

Recall a boundary condition for dielectric interfaces is:

\[ D_{1n}(\vec{r}) = D_{2n}(\vec{r}) \]
We can apply this boundary condition as well! Recall:

\[ \mathbf{D}(\vec{r}) = \varepsilon \mathbf{E}(\vec{r}) = -\varepsilon\nabla V(\vec{r}) \]

where for this case:

\[ \nabla V_1(\vec{r}) = \frac{\partial V_1(z)}{\partial z} \hat{a}_z = C_{1a} \hat{a}_z \]

and

\[ \nabla V_2(\vec{r}) = \frac{\partial V_2(z)}{\partial z} \hat{a}_z = C_{2a} \hat{a}_z \]

Therefore:

\[ \mathbf{D}_1(\vec{r}) = -\varepsilon_1 C_{1a} \hat{a}_z \]

\[ \mathbf{D}_2(\vec{r}) = -\varepsilon_2 C_{2a} \hat{a}_z \]

Since \( \hat{a}_n = \hat{a}_z \), we find from this boundary condition that:

\[ \mathbf{D}_{1n}(\vec{r}) = \mathbf{D}_{2n}(\vec{r}) \]

\[ -\varepsilon_1 C_{1a} = -\varepsilon_2 C_{2a} \]

Now we can solve for our constants! Recall our four equations are:
1) \( V_0 = -C_{1a}(d_1 + d_2) + C_{1b} \) \( \text{from } V_1(z = -(d_1 + d_2)) = V_0 \)

2) \( C_{2b} = 0 \) \( \text{from } V_2(z = 0) = 0 \)

3) \( -C_{1a}d_2 + C_{1b} = -C_{2a}d_2 \) \( \text{from } V_1(z = -d_2) = V_2(z = -d_2) \)

4) \( \varepsilon_1C_{1a} = \varepsilon_2C_{2a} \) \( \text{from } D_{1n}(\vec{r}) = D_{2n}(\vec{r}) \)

Solving these equations, we find (in addition to \( C_{2b}=0 \)):

\[
C_{1a} = \frac{-V_0 \varepsilon_2}{\varepsilon_1 d_2 + \varepsilon_2 d_1}
\]

and

\[
C_{1b} = \frac{V_0 (\varepsilon_1 - \varepsilon_2) d_2}{\varepsilon_1 d_2 + \varepsilon_2 d_1}
\]

and

\[
C_{2a} = \frac{-V_0 \varepsilon_1}{\varepsilon_1 d_2 + \varepsilon_2 d_1}
\]
Inserting these results into the equations:

\[ V_1(z) = C_{1a} z + C_{1b} \]

\[ V_2(z) = C_{2a} z + C_{2b} \]

provides the electric potential field in each region within the parallel plates. From this result, we can determine the electric field and the electric flux density within each region, as well as the charge density on each plate.