Example: The Electric Dipole

Consider two point charges \( (Q_1 \text{ and } Q_2) \), each with equal magnitude but opposite sign, i.e.:

\[
Q_1 = Q \quad \text{and} \quad Q_2 = -Q \quad \text{so} \quad Q_1 = -Q_2
\]

Say these two charges are located on the \( z \)-axis, and separated by a distance \( d \).

The location of charge \( Q_1 = -Q \) is therefore specified by:

position vector \( \vec{r}_1' = \frac{d}{2} \hat{a}_z \)

The location of charge \( Q_2 = -Q \) is therefore specified by:

position vector \( \vec{r}_2' = -\frac{d}{2} \hat{a}_z \)
We call this charge configuration an electric dipole. Note the total charge in a dipole is zero (i.e., $Q_1 + Q_2 = Q - Q = 0$). But, since the charges are located at different positions, the electric field that is created is not zero!

**Q:** Just what is the electric field created by an electric dipole?

**A:** One approach is to use Coulomb’s Law, and add the resulting electric vector fields from each charge together.

However, let’s try a different approach. Let’s find the electric potential field resulting from an electric dipole. We can then take the gradient to find the electric field!

Note that this should be relatively straightforward! We already know the electric potential resulting from a single point charge—the electric potential resulting from two point charges is simply the summation of each:

$$V(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r})$$

where the electric potential $V_1(\vec{r})$, created by charge $Q_1$, is:

$$V_1(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_1|} = \frac{Q}{4\pi\varepsilon_0 \left|\vec{r} - \frac{d}{2}\hat{a}_z\right|}$$
and electric potential $V_2(\vec{r})$, created by charge $Q_2$, is:

$$V_2(\vec{r}) = \frac{Q_2}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_2|} = \frac{-Q}{4\pi\varepsilon_0 |\vec{r} + \frac{d}{2}\hat{a}_z|}$$

Therefore the total electric potential field is:

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 |\vec{r} - \frac{d}{2}\hat{a}_z|} - \frac{Q}{4\pi\varepsilon_0 |\vec{r} + \frac{d}{2}\hat{a}_z|}$$

$$= \frac{Q}{4\pi\varepsilon_0} \left( \frac{1}{|\vec{r} - \frac{d}{2}\hat{a}_z|} - \frac{1}{|\vec{r} + \frac{d}{2}\hat{a}_z|} \right)$$

If the point denoted by $\vec{r}$ is a significant distance away from the electric dipole (i.e., $|\vec{r}| \gg d$), we can use the following approximations:

$$\frac{1}{|\vec{r} - \frac{d}{2}\hat{a}_z|} \approx \frac{1}{|\vec{r}|} + \frac{d \cos \theta}{2|\vec{r}|} = \frac{1}{r} + \frac{d \cos \theta}{2r^2}$$

$$\frac{1}{|\vec{r} + \frac{d}{2}\hat{a}_z|} \approx \frac{1}{|\vec{r}|} - \frac{d \cos \theta}{2|\vec{r}|} = \frac{1}{r} - \frac{d \cos \theta}{2r^2}$$

where $r$ and $\theta$ are the **spherical coordinate** variables of the point denoted by $\vec{r}$. 
Therefore, we find:

\[
V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0} \left( \frac{1}{|\vec{r} - \frac{d}{2}\hat{a}_z|} - \frac{1}{|\vec{r} + \frac{d}{2}\hat{a}_z|} \right)
\]

\[
= \frac{Q}{4\pi\varepsilon_0} \left( \left( \frac{1}{r} + \frac{d \cos \theta}{2r^2} \right) - \left( \frac{1}{r} - \frac{d \cos \theta}{2r^2} \right) \right)
\]

\[
= \frac{Q}{4\pi\varepsilon_0} \frac{d \cos \theta}{r^2}
\]

Note the result. The electric potential field produced by an electric dipole, when centered at the origin and aligned with the z-axis is:

\[
V(\vec{r}) = \frac{Qd \cos \theta}{4\pi\varepsilon_0 \ r^2}
\]

Q: But the original question was, what is the electric field produced by an electric dipole?

A: Easily determined! Just take the gradient of the electric potential function, and multiply by -1.
\[ E(\vec{r}) = -\nabla V(\vec{r}) \]
\[ = -\nabla \left( \frac{Q d}{4\pi\varepsilon_0} \frac{\cos\theta}{r^2} \right) \]
\[ = -\frac{Q d}{4\pi\varepsilon_0} \left[ \cos\theta \frac{d}{dr} \left( \frac{1}{r^2} \right) \hat{a}_r + \frac{1}{r^3} \frac{d (\cos\theta)}{d\theta} \hat{a}_\theta \right] \]
\[ = -\frac{Q d}{4\pi\varepsilon_0} \left[ \left( \frac{-2 \cos\theta}{r^3} \right) \hat{a}_r - \frac{\sin\theta}{r^3} \hat{a}_\theta \right] \]

The static electric field produced by an electric dipole, when centered at the origin and aligned with the \( z \)-axis is:

\[ E(\vec{r}) = \frac{Q d}{4\pi\varepsilon_0} \frac{1}{r^3} \left[ 2\cos\theta \hat{a}_r + \sin\theta \hat{a}_\theta \right] \]

Yikes! *Contrast* this with the electric field of a single point charge. The electric dipole produces an electric field that:

1) Is proportional to \( r^{-3} \) (as opposed to \( r^{-2} \)).

2) Has vector components in both the \( \hat{a}_r \) and \( \hat{a}_\theta \) directions (as opposed to just \( \hat{a}_r \)).

In other words, the electric field does not point away from the electric dipole!
The electric potential produced by an electric dipole looks like:
And the electric field produced by the electric dipole is: