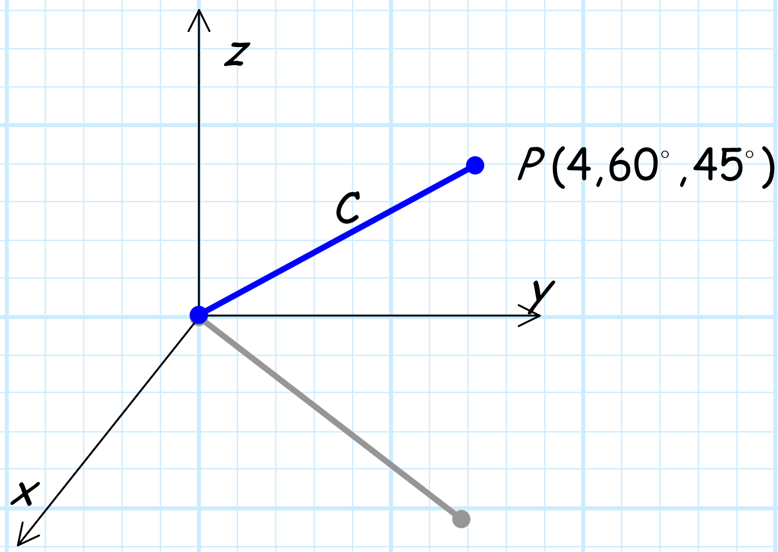
Example: The Line Integral

Consider the vector field:

$$\mathbf{A}(\bar{r}_c) = z\ \hat{a}_x - x\ \hat{a}_y$$

Integrate this vector field over **contour** C, a straight line that begins at the **origin** and ends at point $P\left(r = 4, \theta=60^\circ, \phi=45^\circ\right)$.



Step 1: Determine the **two equalities**, **one inequality**, and proper $\overline{d\ell}$ for the **contour** C.

This contour is formed as the coordinate r changes from r=0 to r=4, where $\theta = 60^\circ$ and $\phi = 45^\circ$ for all points. The two equalities and one inequality that define this contour are thus:

$$0 \le r \le 4 \qquad \theta = 60^\circ \qquad \phi = 45^\circ$$

and the **differential** displacement vector for this contour is therefore:

$$\overline{d\ell} = \overline{dr} = \hat{\mathbf{a}}_r \, dr$$

Step 2: Evaluate the **dot product** $\mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$.

$$\begin{aligned} \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} &= \left(z \ \hat{a}_x - x \ \hat{a}_y\right) \cdot \hat{a}_r \, dr \\ &= \left(z \ \hat{a}_x \cdot \hat{a}_r - x \ \hat{a}_y \cdot \hat{a}_r\right) dr \\ &= \left(z \ \sin\theta \cos\phi - x \ \sin\theta \sin\phi\right) dr \end{aligned}$$

Step 3: Transform all coordinates of the resulting **scalar** field to the **same** system as C.

The contour is a **spherical** contour. Recall that $z = r\cos\theta$ and $x = r\sin\theta\cos\phi$, therefore:

$$\begin{aligned} \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} &= \left(z \ \sin\theta \cos\phi - x \ \sin\theta \sin\phi\right) dr \\ &= \left(r\cos\theta \ \sin\theta\cos\phi - r\sin\theta\cos\phi \ \sin\theta\sin\phi\right) dr \\ &= r \ \sin\theta\cos\phi\left(\cos\theta - \sin\theta\sin\phi\right) dr \end{aligned}$$

Step 4: Evaluate the scalar field using the **two** coordinate **equalities** that describe contour C.

Recall that θ=60° and ϕ=45° at **every** point along the contour we are integrating over. Thus, functions of θ or ϕ are **constants** with respect to the integration! For example, $\cos\theta = \cos 45^\circ = 0.5$. Therefore:

$$\begin{aligned}\mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} &= r\ \sin 60^\circ \cos 45^\circ \left(\cos 60^\circ - \sin 60^\circ \sin 45^\circ\right) dr \\ &= r\ \sqrt{3/4}\ \sqrt{1/2}\left(1/2 - \sqrt{3/4}\sqrt{1/2}\right) dr \\ &= r\sqrt{3/8}\left(\frac{\sqrt{2}-\sqrt{3}}{\sqrt{8}}\right) dr \\ &= \left(\frac{\sqrt{6}-3}{8}\right) r\ dr\end{aligned}$$

Step 5: Determine the **limits of integration** from the **inequality** that describes contour *C (be careful of order!)*.

We note the contour is described as:

$$0 \le r \le 4$$

and the contour C moves from $r = 0$ to $r = 4$. Thus, we integrate from 0 to 4 :

$$\int_C \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} = \int_0^4 \left(\frac{\sqrt{6}-3}{8}\right) r\ dr$$

Note: if the contour ran from $r = 4$ to $r = 0$ the limits of integration would be **flipped**! I.E.,

$$\int_4^0 \left(\frac{\sqrt{6}-3}{8}\right) r\ dr$$

It is readily apparent that the line integral from $r = 0$ to $r = 4$ is the opposite (i.e., **negative**) of the integral from $r = 4$ to $r = 0$.

Step 6: Integrate the remaining function of **one** coordinate variable.

$$
\begin{aligned}
\int_C \mathbf{A}(\overline{r_c}) \cdot \overline{d\ell} &= \int_0^4 \left(\frac{\sqrt{6}-3}{8}\right) r\, dr \\
&= \left(\frac{\sqrt{6}-3}{8}\right) \int_0^4 r\, dr \\
&= \left(\frac{\sqrt{6}-3}{8}\right)\left(\frac{4^2}{2} - \frac{0^2}{2}\right) \\
&= \sqrt{6} - 3
\end{aligned}
$$