Example: The Volume Integral

Let's evaluate the **volume** integral:

$$\iiint_V g(\bar{r})\, dv$$

where $g(\bar{r}) = 1$ and the volume V is a **sphere** with radius R.

In other words, the volume V is described as:

$$0 \le r \le R$$

$$0 \le \theta \le \pi$$

$$0 \le \phi \le 2\pi$$

And thus we use for the **differential** volume *dv:*

$$dv = \overline{dr} \cdot \overline{d\theta} \times \overline{d\phi} = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Therefore:

$$\iiint_V g(\bar{r})\, dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{R} r^2 dr$$

$$= 2\pi(2)\frac{R^3}{3}$$

$$= \frac{4}{3}\pi R^3$$

Hey look! The answer is the **volume** (e.g., in m^3) of a **sphere**!

Now, this result provided the numeric volume of V **only** because $g(\bar{r}) = 1$. We find that the total volume of **any** space V can be determined this way:

$$\text{Volume of } V = \iiint_V (1)\, dv$$

Typically though, we find that $g(\bar{r}) \neq 1$, and thus the volume integral does **not** provide the numeric volume of space V.

Q: *So what's the volume integral even good for?*

A: Generally speaking, the scalar function $g(\bar{r})$ will be a density function, with units of **things/unit volume**. Integrating $g(\bar{r})$ with the volume integral provides us the **number of things** within the space V!

For example, let's say $g(\bar{r})$ describes the **density** of a big **swarm of insects**, using units of ***insects/m³*** (i.e., insects are the **things**). Note that $g(\bar{r})$ must indeed a **function** of position, as the density of insects changes at different locations throughout the swarm.

Now say we want to know the total number of insects within the swarm, which occupies some space *V*. We can determine this by simply applying the volume integral!

$$\text{number of insects in swarm } = \iiint_V g(\bar{r})\, dv$$

where space *V* completely encloses the insect swarm.