Example: The Volume Integral

Let's evaluate the volume integral:

\[ \iiint_V g(\vec{r}) \, dV \]

where \( g(\vec{r}) = 1 \) and the volume \( V \) is a sphere with radius \( R \).

In other words, the volume \( V \) is described as:

\[
\begin{align*}
0 &\leq r \leq R \\
0 &\leq \theta \leq \pi \\
0 &\leq \phi \leq 2\pi
\end{align*}
\]

And thus we use for the differential volume \( dV \):

\[ dV = dr \cdot d\theta \times d\phi = r^2 \sin \theta \, dr \, d\theta \, d\phi \]

Therefore:
\[
\iiint_V g(\vec{r}) \, dV = \int_0^{2\pi} \int_0^\pi \int_0^R r^2 \sin \theta \, dr \, d\theta \, d\phi \\
= \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^R r^2 dr \\
= 2\pi(2) \frac{R^3}{3} \\
= \frac{4}{3} \pi R^3
\]

Hey look! The answer is the volume (e.g., in m\(^3\)) of a sphere!

Now, this result provided the numeric volume of \( V \) only because \( g(\vec{r}) = 1 \). We find that the total volume of any space \( V \) can be determined this way:

\[
\text{Volume of } V = \iiint_V (1) \, dV
\]

Typically though, we find that \( g(\vec{r}) \neq 1 \), and thus the volume integral does not provide the numeric volume of space \( V \).

Q: So what's the volume integral even good for?

A: Generally speaking, the scalar function \( g(\vec{r}) \) will be a density function, with units of things/unit volume. Integrating \( g(\vec{r}) \) with the volume integral provides us the number of things within the space \( V \)!
For example, let's say $g(\mathbf{r})$ describes the density of a big swarm of insects, using units of $\text{insects/m}^3$ (i.e., insects are the things). Note that $g(\mathbf{r})$ must indeed a function of position, as the density of insects changes at different locations throughout the swarm.

Now say we want to know the total number of insects within the swarm, which occupies some space $V$. We can determine this by simply applying the volume integral!

\[
\text{number of insects in swarm} = \iiint_V g(\mathbf{r}) \, dV
\]

where space $V$ completely encloses the insect swarm.