Example: Vector Algebra

Consider the **scalar** expression:

$$ac + bc + bd + ad$$

We can manipulate and simply this expression using the rules of **scalar algebra**:

$$\begin{aligned} ac + bc + bd + ad &= ac + ad + bc + bd && \text{(commutative)} \\ &= (ac + ad) + (bc + bd) && \text{(associative)} \\ &= a(c + d) + b(c + d) && \text{(distributive)} \\ &= (a + b)(c + d) && \text{(distributive)} \end{aligned}$$

We can likewise perform a similar analysis on vector expressions! Consider now the expression:

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} \times \mathbf{A}$$

We can show that this is actually a very familiar and basic vector operation!

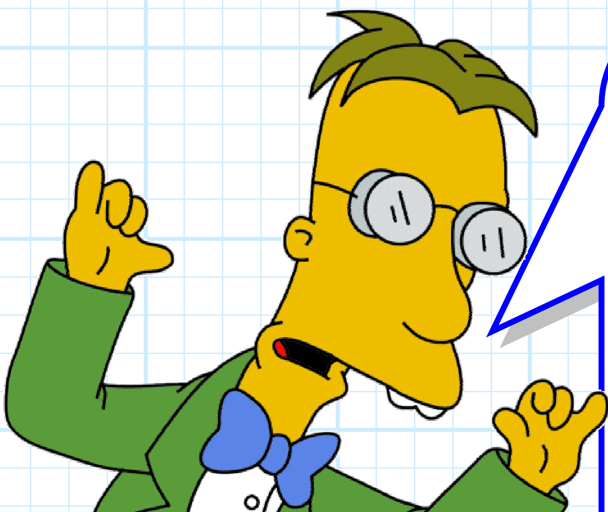
$$\begin{aligned} (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} \times \mathbf{A} &= \mathbf{C} \cdot \mathbf{A} \times (\mathbf{A} + \mathbf{B}) && \text{(Triple product identity)} \\ &= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}) && \text{(Cross Product Distibutive)} \\ &= \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} && (\text{Since } \mathbf{A} \times \mathbf{A} = 0) \\ &= \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} && \text{(Triple product identity)} \end{aligned}$$

Or, for example, if we consider:

$$(\mathbf{A}+\mathbf{B})\cdot(\mathbf{B}+2\mathbf{A})$$

we find:

$$
\begin{aligned}
&= (\mathbf{A}+\mathbf{B})\cdot(\mathbf{B}+2\mathbf{A}) \\
&= \mathbf{A}\cdot(\mathbf{B}+2\mathbf{A})+\mathbf{B}\cdot(\mathbf{B}+2\mathbf{A}) && \text{(dot product distributive)} \\
&= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot 2\mathbf{A}+\mathbf{B}\cdot\mathbf{B}+\mathbf{B}\cdot 2\mathbf{A} && \text{(dot product distributive)} \\
&= \mathbf{A}\cdot\mathbf{B}+2\,\mathbf{A}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}+2\,\mathbf{B}\cdot\mathbf{A} && \text{(scalar multiply commutative)} \\
&= \mathbf{A}\cdot\mathbf{B}+2|\mathbf{A}|^2+|\mathbf{B}|^2+2\,\mathbf{B}\cdot\mathbf{A} && (\mathbf{C}\cdot\mathbf{C}=|\mathbf{C}|^2 \text{ identity}) \\
&= \mathbf{A}\cdot\mathbf{B}+2|\mathbf{A}|^2+|\mathbf{B}|^2+2\,\mathbf{A}\cdot\mathbf{B} && \text{(dot product communitive)} \\
&= 2|\mathbf{A}|^2+2\,\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{B}+|\mathbf{B}|^2 && \text{(vector addition commutative)} \\
&= 2|\mathbf{A}|^2+(2+1)\,\mathbf{A}\cdot\mathbf{B}+|\mathbf{B}|^2 && \text{(scalar multiply distributive)} \\
&= |\mathbf{A}|^2+3\mathbf{A}\cdot\mathbf{B}+|\mathbf{B}|^2 && (2+1=3)
\end{aligned}
$$



Keep in mind one very important point when doing vector algebra—the expression ***can never change type*** *(e.g., from vector to scalar)!*

In other words, if the expression ***initially*** *results in a vector (or scalar), then after each manipulation, the result must* ***also*** *be a vector (or scalar).*

For example, we find that the following expression **cannot** possibly be true!

$$\mathbf{A}\times\mathbf{B}+(\mathbf{B}\cdot\mathbf{C})\mathbf{A}=\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C})+(\mathbf{A}+\mathbf{B})\cdot\mathbf{C}$$

Q: *Do **you** see why?*

A: ____

*Likewise, be careful **not** to create expressions that have **no** mathematical meaning whatsoever!*

Examples include:

$$(\mathbf{A}\cdot\mathbf{B})\times\mathbf{C}$$

$$\mathbf{A}+(\mathbf{B}\cdot\mathbf{C})$$