Helmholtz's Theorems

Consider a differential equation of the following form:

\[ g(t) = \frac{df(t)}{dt} \]

where \( g(t) \) is an explicit known function, and \( f(t) \) is the unknown function that we seek.

For example, the differential equation:

\[ 3t^2 + t - 1 = \frac{df(t)}{dt} \]

has a solution:

\[ f(t) = t^3 + \frac{t^2}{2} - t + c \]

Thus, the derivative of \( f(t) \) provides sufficient knowledge to determine the original function \( f(t) \) (to within a constant).

An interesting question, therefore, is whether knowledge of the divergence and or curl of a vector field is sufficient to determine the original vector field.
For example, say we don't know the expression for vector field \(A(\vec{r})\), but we do know its divergence is some scalar function \(g(\vec{r})\):

\[
\nabla \cdot A(\vec{r}) = g(\vec{r})
\]

Can we, then, determine the vector field \(A(\vec{r})\)? For example, can \(A(\vec{r})\) be determined from the expression:

\[
\nabla \cdot A(\vec{r}) = x(y^2 - z^3)
\]

On the other hand, perhaps the knowledge of the curl is sufficient to find \(A(\vec{r})\), i.e.:

\[
\nabla \times A(\vec{r}) = \cos \frac{z\pi}{y} \hat{a}_x + (x^2 - 6) \hat{a}_y + e^{-\frac{x}{y}} \hat{a}_z
\]

therefore \(A(\vec{r})=?????

It turns out that neither the knowledge of the divergence nor the knowledge of the curl alone is sufficient to determine a vector field. However, knowledge of both the curl and divergence of a vector field is sufficient!

Take this tip from me!

If you know \(\nabla \cdot A(\vec{r})\) and you know \(\nabla \times A(\vec{r})\), you have enough information to determine the vector field \(A(\vec{r})\)!
Q: But why do we need knowledge of both the divergence and curl of a vector field in order to determine the vector field?

A: I know the answer to that as well!

It's because every vector field can be written as the sum of a conservative field and a solenoidal field!

That's correct! Any and every possible vector field \( \mathbf{A}(\mathbf{r}) \) can be expressed as the sum of a conservative field \( \mathbf{C}_A(\mathbf{r}) \) and a solenoidal field \( \mathbf{S}_A(\mathbf{r}) \):

\[
\mathbf{A}(\mathbf{r}) = \mathbf{C}_A(\mathbf{r}) + \mathbf{S}_A(\mathbf{r})
\]

Note then if \( \mathbf{C}_A(\mathbf{r}) = 0 \), the vector field \( \mathbf{A}(\mathbf{r}) = \mathbf{S}_A(\mathbf{r}) \) is solenoidal. Likewise, if \( \mathbf{S}_A(\mathbf{r}) = 0 \) the vector field \( \mathbf{A}(\mathbf{r}) = \mathbf{C}_A(\mathbf{r}) \) is conservative.

Of course, if neither term is zero (i.e., \( \mathbf{C}_A(\mathbf{r}) \neq 0 \) and \( \mathbf{S}_A(\mathbf{r}) \neq 0 \)), the vector field \( \mathbf{A}(\mathbf{r}) \) is neither conservative nor solenoidal!
Consider then what happens when we take the divergence of a vector field \( \mathbf{A}(\mathbf{r}) \):

\[
\nabla \cdot \mathbf{A}(\mathbf{r}) = \nabla \cdot \mathbf{C}_A(\mathbf{r}) + \nabla \cdot \mathbf{S}_A(\mathbf{r})
\]

\[
= \nabla \cdot \mathbf{C}_A(\mathbf{r}) + 0
\]

\[
= \nabla \cdot \mathbf{C}_A(\mathbf{r})
\]

Look what happened! Since the divergence of a solenoidal field is zero, the divergence of a general vector field \( \mathbf{A}(\mathbf{r}) \) really just tells us the divergence of its conservative component.

The divergence of a vector field tells us nothing about its solenoidal component \( \mathbf{S}_A(\mathbf{r}) \)!

Thus, from \( \nabla \cdot \mathbf{A}(\mathbf{r}) \) we can determine \( \mathbf{C}_A(\mathbf{r}) \), but we haven't a clue about what \( \mathbf{S}_A(\mathbf{r}) \) is!

Likewise, the curl of \( \mathbf{A}(\mathbf{r}) \) is:

\[
\nabla \times \mathbf{A}(\mathbf{r}) = \nabla \times \mathbf{C}_A(\mathbf{r}) + \nabla \times \mathbf{S}_A(\mathbf{r})
\]

\[
= 0 + \nabla \times \mathbf{S}_A(\mathbf{r})
\]

\[
= \nabla \times \mathbf{S}_A(\mathbf{r})
\]

Look what happened! Since the curl of a conservative field is zero, the curl of a general vector field \( \mathbf{A}(\mathbf{r}) \) really just tells us the curl of its solenoidal component.

The curl of a vector field tells us nothing about its conservative component \( \mathbf{C}_A(\mathbf{r}) \)!
Thus, from $\nabla \times \mathbf{A}(\mathbf{r})$ we can determine $\mathbf{S}_A(\mathbf{r})$, but we haven't a clue about what $\mathbf{C}_A(\mathbf{r})$ is!

**CONCLUSION:** We require knowledge of both $\nabla \cdot \mathbf{A}(\mathbf{r})$ (for $\mathbf{C}_A(\mathbf{r})$) and $\nabla \times \mathbf{A}(\mathbf{r})$ (for $\mathbf{S}_A(\mathbf{r})$) to determine the vector field $\mathbf{A}(\mathbf{r})$.

From a physical standpoint, this makes perfect sense!

Recall that we determined the curl $\nabla \times \mathbf{A}(\mathbf{r})$ identifies the rotational sources of vector field $\mathbf{A}(\mathbf{r})$, while the divergence $\nabla \cdot \mathbf{A}(\mathbf{r})$ identifies the divergent (or convergent) sources.

Once we know the sources of vector field $\mathbf{A}(\mathbf{r})$, we can of course find vector field $\mathbf{A}(\mathbf{r})$.

Q: Exactly how do we find $\mathbf{A}(\mathbf{r})$ from its sources ($\nabla \cdot \mathbf{A}(\mathbf{r})$ and $\nabla \times \mathbf{A}(\mathbf{r})$)??

A1: I don’t know.
**A2:** Note the **sources** of a vector field are determined from **derivative** operations (i.e., divergence and curl) on the vector field.

We can therefore conclude that a vector field \( \mathbf{A}(\mathbf{r}) \) can be determined from its sources with **integral** operations!

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We'll learn much more about integrating sources later in the course!
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