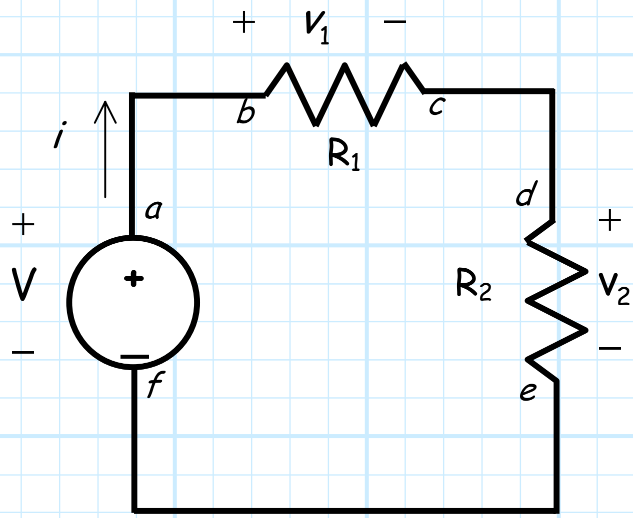
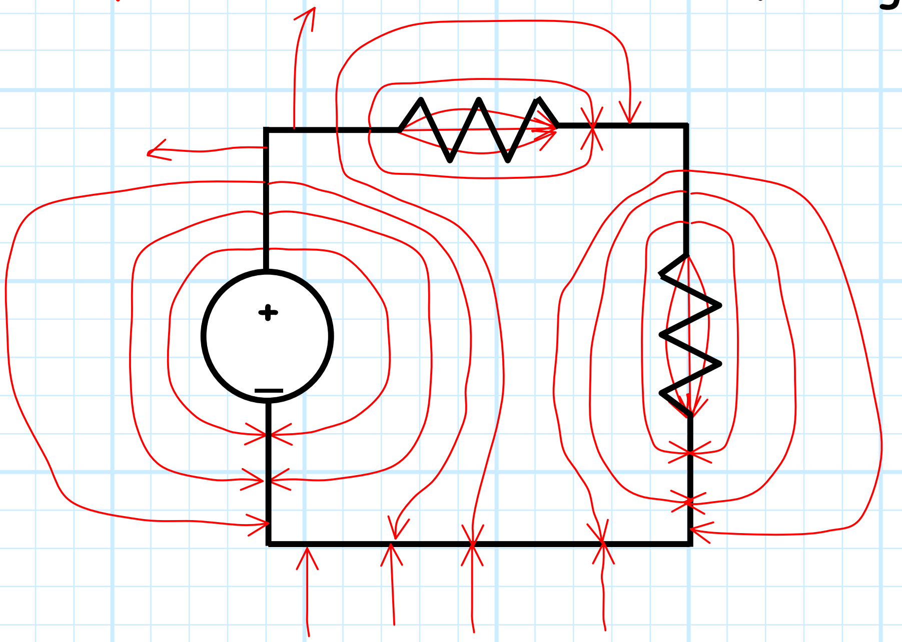
Kirchoff's Voltage Law

Consider a simple electrical **circuit**:



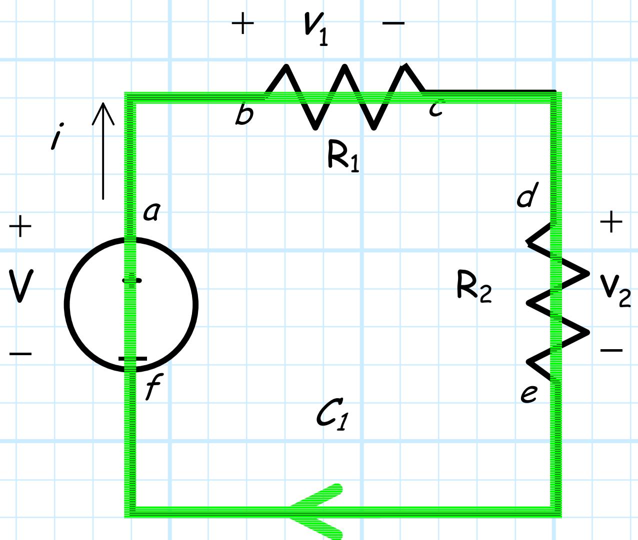
We find that if the **voltage source** is on (i.e., $V \neq 0$), then there will be electric potential differences (i.e., voltage) between different points of the circuit. This can **only** be true if **electric fields** are present!

The electric field in this circuit will "look" **something** like this:



So, instead of using circuit theory, let's use our new **electromagnetic** knowledge to **analyze** this circuit.

First, consider a **contour** C_1 that follows the circuit path.



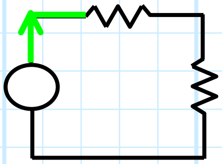
Using this path, let's **evaluate** the contour integral:

$$\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell}$$

This is most easily done by breaking the contour C_1 into **six sections**: section 1 extends from point *a* to point *b*, section 2 extends from point *b* to point *c*, etc. Thus, the integral becomes:

$$\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = \int_a^b \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_b^c \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_c^d \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_d^e \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_e^f \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_f^a \mathbf{E}(\bar{r}) \cdot \overline{d\ell}$$

Let's evaluate each term **individually**:

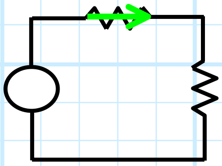


Section 1 (*a* to *b*)

In this section, the contour follows the **wire** from the voltage source to the first resistor. We know that the electric field in a perfect conductor is **zero**, and likewise in a good conductor it is **very small**. Assuming the wire is in fact made of a **good conductor** (e.g. copper), we can approximate the electric field **within** the wire (and thus at **every** point along section 1) as **zero** (i.e., $\mathbf{E}(\bar{r}) = 0$). Therefore, this first integral equals zero!

$$\int_a^b \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$

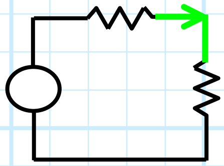
This of course makes sense! We know that the electric potential difference across a **wire** is **zero volts**.



Section 2 (*b* to *c*)

In this section, the contour moves through the first **resistor**. The contour integral along this section therefore allows us to determine the electric **potential difference** across this resistor. Let's denote this potential difference as v_1 :

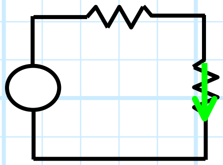
$$\int_b^c \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = V(P_b) - V(P_c) = v_1$$



Section 3 (*c* to *d*)

Just like section 1, the contour follows a **wire**, and thus the electric field along this section of the contour is **zero**, as is the potential difference between point *c* and point *d*.

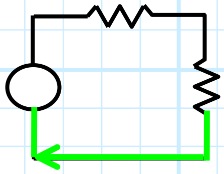
$$\int_c^d \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$



Section 4 (*d* to *e*)

Just like section 2, the contour moves through a **resistor**. The contour integral for this section is thus equal to the potential difference across this **second** resistor, which we denote as v_2:

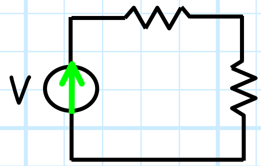
$$\int_d^e \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = V(P_d) - V(P_e) = v_2$$



Section 5 (*e* to *f*)

Again, the contour follows a conducting **wire**—and again, the electric field along the contour and the potential difference across it are both **zero**:

$$\int_e^f \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$



Section 6 (f to a)

This **final** section of contour C_1 extends through the **voltage source**, thus the contour integral of this section provides the electric potential difference between the two terminals of the this voltage source (i.e., $V(P_f) - V(P_a)$). By **definition**, the potential difference between points a and f is a value of V volts (i.e., $V(P_a) - V(P_f) = \text{V}$). Therefore, we find that the contour integral of section 6 is :

$$\begin{aligned}\int_f^a \mathbf{E}(\bar{r}) \cdot \overline{d\ell} &= V(P_f) - V(P_a) \\ &= -\left(V(P_a) - V(P_f)\right) \\ &= -\text{V}\end{aligned}$$

Whew! Now let's **combine** these results to determine the contour integral for the **entire** contour C_1.

$$\begin{aligned}\oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = &\int_a^b \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_b^c \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_c^d \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \\ &\int_d^e \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_e^f \mathbf{E}(\bar{r}) \cdot \overline{d\ell} + \int_f^a \mathbf{E}(\bar{r}) \cdot \overline{d\ell} \\ = &\, 0 + v_1 + 0 + v_2 + 0 - V \\ = &\, v_1 + v_2 - V\end{aligned}$$

Q: *Wait; I've forgotten, Why are we evaluating these contour integrals ?*

A: Remember, since the electric field is **static**, we also know that integral around any closed contour is **zero**. Thus, we can conclude that:

$$0 = \oint_{C_1} \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = v_1 + v_2 - V$$

In other words, we find by performing an **electromagnetic** analysis of the circuit, the voltages across each circuit element are related as:

$$v_1 + v_2 - V = 0$$

Q: *You **have** wasted my time! Using **only** Kirchoff's Voltage Law (KVL), **I** arrived at **precisely** the same result ($v_1 + v_2 - V = 0$). **I** think the above equation is true because of **KVL**, not because of your **fancy** electromagnetic theory!*

A: It **is** true that the result we obtained by integrating the electric field around the circuit contour is **likewise** apparent from **KVL**. However, this result is **still** attributable to electrostatic physics, because KVL is a **direct** result of electrostatics!

The electrostatic equation :

$$\oint_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = 0$$

when applied to the closed contour of any **circuit**, results in **Kirchoff's Voltage Law**, i.e.:

$$\sum_n v_n = 0$$

where v_n are the electric potential differences across each element of a circuit "loop" (i.e., closed contour).

Gustav Robert Kirchhoff (1824-1887), German physicist, announced the laws that allow calculation of the currents, voltages, and resistances of electrical networks in 1845, when he was only **twenty-one**! His other work established the technique of spectrum analysis that he applied to determine the composition of the Sun.

From www.ee.umd.edu/~taylor/frame5.htm