Maxwell’s Equations (Yet Again)

Now let’s go back and again examine Maxwell’s Equations, which we first looked at in Chapter 3:

\[ \nabla \times \mathbf{B}(\bar{r},t) = \mu_0 \mathbf{J}(\bar{r},t) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\bar{r},t)}{\partial t} \]

\[ \nabla \times \mathbf{E}(\bar{r},t) = -\frac{\partial \mathbf{B}(\bar{r},t)}{\partial t} \]

\[ \nabla \cdot \mathbf{E}(\bar{r},t) = \frac{\rho_\varepsilon(\bar{r},t)}{\varepsilon_0} \]

\[ \nabla \cdot \mathbf{B}(\bar{r},t) = 0 \]

Now that we have introduced the concept of dielectrics and magnetic material, we can write these equations more generally as:
\[ \nabla \times \mathbf{H}(\vec{r}, t) = \mathbf{J}(\vec{r}, t) + \frac{\partial \mathbf{D}(\vec{r}, t)}{\partial t} \]

\[ \nabla \times \mathbf{E}(\vec{r}, t) = -\frac{\partial \mathbf{B}(\vec{r}, t)}{\partial t} \]

\[ \nabla \cdot \mathbf{D}(\vec{r}, t) = \rho_v(\vec{r}, t) \]

\[ \nabla \cdot \mathbf{B}(\vec{r}, t) = 0 \]

These are the point form of Maxwell's Equations; we can also write them in integral form:

\[ \oint_c \mathbf{H}(\vec{r}, t) \cdot d\ell = I_{enc} + \int_s \frac{\partial \mathbf{D}(\vec{r}, t)}{\partial t} \cdot ds \]

\[ \oint_c \mathbf{E}(\vec{r}, t) \cdot d\ell = -\int_s \frac{\partial \mathbf{B}(\vec{r}, t)}{\partial t} \cdot ds \]

\[ \oiint_s \mathbf{D}(\vec{r}, t) \cdot ds = Q_{enc} \]

\[ \oiint_s \mathbf{B}(\vec{r}, t) \cdot ds = 0 \]
But, we have a **problem**! Maxwell’s Equations now (i.e., in material) has too many unknowns and too few equations!

To complete our electromagnetic knowledge, we must consider the constitutive equations, which are dependent on the **material properties**:

\[
\mathbf{D}(\vec{r}) = \varepsilon \mathbf{E}(\vec{r}) \\
\mathbf{B}(\vec{r}) = \mu \mathbf{H}(\vec{r}) \\
\mathbf{J}(\vec{r}) = \sigma \mathbf{E}(\vec{r})
\]

Now, let’s consider again Maxwell’s equations:

\[
\begin{align*}
\nabla \times \mathbf{H}(\vec{r}, t) &= \mathbf{J}(\vec{r}, t) + \frac{\partial \mathbf{D}(\vec{r}, t)}{\partial t} \\
\nabla \times \mathbf{E}(\vec{r}, t) &= -\frac{\partial \mathbf{B}(\vec{r}, t)}{\partial t} \\
\nabla \cdot \mathbf{D}(\vec{r}, t) &= \rho_{\nu}(\vec{r}, t) \\
\nabla \cdot \mathbf{B}(\vec{r}, t) &= 0
\end{align*}
\]

We can interpret these equations as relating **sources** and the **fields** these sources create. The **sources** appear on right side of Maxwell’s equations, whereas the **fields** appear on the left.
For example, we know that an electric field and electric flux density is created from charge:

\[ \nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho_v(\mathbf{r}, t) \]

\[ \mathbf{D}(\mathbf{r}) = \varepsilon \mathbf{E}(\mathbf{r}) \]

But, we also know that an electric field and electric flux density can be created (induced) by a time varying magnetic flux density:

\[ \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \]

\[ \mathbf{D}(\mathbf{r}) = \varepsilon \mathbf{E}(\mathbf{r}) \]

Likewise, we know that current is the source of a magnetic field and magnetic flux density:

\[ \nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) \]

\[ \mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r}) \]

But, note we have one source left! Note that it appears that a time-varying electric flux density can “induce” a magnetic field, much in the same way that a time-varying magnetic flux density induces and electric field.
\[ \nabla \times \mathbf{H}(\vec{r}, t) = \frac{\partial \mathbf{D}(\vec{r}, t)}{\partial t} \]

\[ \mathbf{B}(\vec{r}) = \mu \mathbf{H}(\vec{r}) \]

Q: What the heck is \( \frac{\partial \mathbf{D}(\vec{r}, t)}{\partial t} \)?

A: Try taking the divergence of Ampere's Law.

\[ \nabla \cdot \nabla \times \mathbf{H}(\vec{r}, t) = \nabla \cdot \mathbf{J}(\vec{r}, t) + \frac{\partial \nabla \cdot \mathbf{D}(\vec{r}, t)}{\partial t} \]

Since we know that the divergence of every curl is zero (i.e., \( \nabla \cdot \nabla \times \mathbf{H}(\vec{r}, t) = 0 \)), we find:

\[ \nabla \cdot \mathbf{J}(\vec{r}, t) = -\frac{\partial \nabla \cdot \mathbf{D}(\vec{r}, t)}{\partial t} \]

Recall that often we find that the divergence of current density \( \mathbf{J}(\vec{r}) \) is zero (i.e., \( \nabla \cdot \mathbf{J}(\vec{r}) = 0 \)), as charge cannot be created or destroyed. The exception is when charge "pile up", or diminish at some point. In this case, the charge density \( \rho_v(\vec{r}) \) must change as a function of time.

Recall that this was expressed as the continuity equation:

\[ \nabla \cdot \mathbf{J}(\vec{r}) = -\frac{\partial \rho_v(\vec{r})}{\partial t} \]
We called this type of current density—whose divergence is not zero—displacement current $J_c(\vec{r})$.

Therefore, we can state:

$$\nabla \cdot J_c(\vec{r}) = -\frac{\partial \rho_v(\vec{r})}{\partial t}$$

But, recall that $\rho_v(\vec{r}) = \nabla \cdot D(\vec{r})$, therefore:

$$\nabla \cdot J_c(\vec{r}) = -\frac{\partial \nabla \cdot D(\vec{r})}{\partial t}$$

Or, more specifically:

$$-\frac{\partial D(\vec{r})}{\partial t} = J_c(\vec{r})$$

= displacement current

Therefore Ampere’s Law can be written as:

$$\nabla \times H(\vec{r}, t) = J(\vec{r}, t) + \frac{\partial D(\vec{r}, t)}{\partial t}$$

$$= J(\vec{r}, t) - J_c(\vec{r}, t)$$

The most important application of displacement current is when considering capacitors. We know that:

$$i(t) = C \frac{dV(t)}{dt}$$
Yet we also know that the conductors of a capacitor are typically separated by a dielectric with almost no conductance ($\sigma \approx 0$). Thus, the current density $\mathbf{J}(\mathbf{r})$ in the dielectric is zero ($\mathbf{J}(\mathbf{r}) = 0$).

**Q:** So how can current $i(t)$ be flowing??

**A:** Displacement current! The charge from current $i(t)$ does not move through the capacitor, but instead “pile up” at each plate. This change in charge density $\rho_s$ at each plate is equivalent to a current—a displacement current.
A capacitor is analogous to the “storage tank” that we discussed in chapter 3.