Maxwell's Equations

Consider what we now know:

1) Law of Charge Conservation

2) Coulomb's Law of Force

3) Ampere's Law of Force

4) Lorentz Force Law

These are all valid laws, but they are not complete. That is, they do not completely describe the relationships between $\mathbf{J}(\mathbf{r})$, $\mathbf{\rho}_v(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$, and $\mathbf{E}(\mathbf{r})$.

In 1873, James Clerk Maxwell published a book on electromagnetics, which included a complete, unified theory.

This theory includes 4 equations relating $\mathbf{J}(\mathbf{r},t)$, $\mathbf{\rho}_v(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, and $\mathbf{E}(\mathbf{r},t)$, called Maxwell's Equations.
\[ \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \]

\[ \nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\rho_v(\mathbf{r}, t)}{\varepsilon_0} \]

\[ \nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{J}(\mathbf{r}, t) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} \]

\[ \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \]

From Helmholtz’s Theorems, we know that we must know both the divergence and the curl of a vector field in order to determine the vector field.

Note Maxwell’s Equation does this for both the electric field \( \mathbf{E}(\mathbf{r}, t) \) and magnetic flux density \( \mathbf{B}(\mathbf{r}, t) \)!

**Q:** Is the magnetic flux density \( \mathbf{B}(\mathbf{r}, t) \) conservative, solenoidal, or neither?

**A:**
* Since the divergence of the magnetic flux density is zero ($\nabla \cdot B(\vec{r},t) = 0$), it is a solenoidal vector field.

* Thus, all the things that we learned about solenoidal fields are true for the magnetic flux density $B(\vec{r},t)$.

* Likewise, the sources of this rotational field appear to be current (i.e., $\mu_0 J(\vec{r},t)$), and/or a time-varying electric field:

$$\nabla \times B(\vec{r},t) = \mu_0 J(\vec{r},t) + \mu_0 \varepsilon_0 \frac{\partial E(\vec{r},t)}{\partial t}$$

Note that permittivity $\varepsilon_0$ and permeability $\mu_0$ of free space appear also in Maxwell's Equations!

**Q:** Is the electric field $E(\vec{r},t)$ conservative, solenoidal, or neither?

**A:**
* Since neither the curl nor the divergence of the electric field is zero, the electric field is neither conservative nor solenoidal.

* Instead, it is apparent that the electric field has both a solenoidal and conservative vector component!

* The source of the solenoidal component of the electric field \( \mathbf{E}(\vec{r},t) \) appears to be a time-varying magnetic flux density:

\[
\nabla \times \mathbf{E}(\vec{r},t) = -\frac{\partial \mathbf{B}(\vec{r},t)}{\partial t}
\]

* Whereas the source of the conservative component of \( \mathbf{E}(\vec{r},t) \) appears to be charge:

\[
\nabla \cdot \mathbf{E}(\vec{r},t) = \frac{\rho_v(\vec{r},t)}{\varepsilon_0}
\]

**Q:** But, what else do Maxwell's Equations mean?

**A:** They mean that the rest of the semester will be very busy!