Poisson's and Laplace's Equation

We know that for the case of static fields, Maxwell's Equations reduces to the **electrostatic** equations:

$$\nabla x \mathbf{E}(\bar{r}) = 0 \quad \text{and} \quad \nabla \cdot \mathbf{E}(\bar{r}) = \frac{\rho_v(\bar{r})}{\varepsilon_0}$$

We can **alternatively** write these equations in terms of the **electric potential field** $V(\bar{r})$, using the relationship $\mathbf{E}(\bar{r}) = -\nabla V(\bar{r})$:

$$-\nabla x \nabla V(\bar{r}) = 0 \quad \text{and} \quad -\nabla \cdot \nabla V(\bar{r}) = \frac{\rho_v(\bar{r})}{\varepsilon_0}$$

Let's examine the **first** of these equations. Recall that we determined in Chapter 2 that:

$$\nabla x \nabla g(\bar{r}) = 0$$

for **any** and **all** scalar functions $g(\bar{r})$. In other words the first equation is simply a **mathematical identity**—it says **nothing** physically about the electric potential field $V(\bar{r})$!

The **second** equation includes the operation $\nabla \cdot \nabla$. We recall from Chapter 2 that this operation is called the scalar **Laplacian:**

$$\nabla \cdot \nabla = \nabla^2$$

Therefore, we can write the relationship between charge density and the electric potential field in terms of one equation!

$$\nabla^2 V(\bar{\mathbf{r}}) = -\frac{\rho_v(\bar{\mathbf{r}})}{\varepsilon_0}$$

This equation is known as **Poisson's Equation**, and is essentially the "Maxwell's Equation" of the electric potential field $V(\bar{\mathbf{r}})$.

Note that for points where **no charge** exist, Poisson's equation becomes:

$$\nabla^2 V(\bar{\mathbf{r}}) = 0$$

This equation is know as **Laplace's Equation.** Although it looks very **simple**, most scalar functions will **not** satisfy Laplace's Equation! Only a **special class** of scalar fields, called **analytic functions** will satisfy Laplace's equation.